

ON FINITELY STABLE DOMAINS

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ABSTRACT.

Among other results, we prove the following:

- (1) An integral domain R is stable and one-dimensional if and only if R is finitely stable and Mori. If R satisfies these two equivalent conditions, then each overring of R also satisfies these conditions and it is 2- v -generated.
- (2) A locally Archimedean stable domain satisfies accp.
- (3) A stable domain R is Archimedean if and only if every nonunit of R belongs to a height-one prime ideal of R' (this result is related to Ohm's Theorem for Prüfer domains).
- (4) An Archimedean stable domain R is one-dimensional if and only if R' is equidimensional (generally, an Archimedean stable local domain is not necessarily one-dimensional).
- (5) An Archimedean finitely stable semilocal domain with stable maximal ideals is locally Archimedean, but generally, neither Archimedean stable domains, nor Archimedean semilocal domains are necessarily locally Archimedean.
- (6) A stable radical ideal is divisorial.

1. INTRODUCTION

In this introduction we start with a short reminder of finitely stable and stable rings, recall the definitions of other classes of rings that we use here, as Mori, Archimedean, etc., and finally summarize our main results. By a ring we mean a commutative ring with unity. A *local ring* is a ring with a unique maximal ideal, not necessarily Noetherian. A *semilocal ring* is a ring with just finitely many maximal ideals.

Motivated by earlier work of H. Bass [4] and J. Lipman [12] on the number of generators of an ideal, in 1972 J. Sally and W. Vasconcelos defined an ideal I of a ring R to be *stable* if I is projective over its endomorphism ring; they called R a *stable ring* if each nonzero ideal of R is stable [32, 33]. Stability of rings is often determined by the stability of regular ideals, that is, ideals containing a nonzero divisor. D. Rush studied the rings such that each finitely generated regular ideal is stable, in particular in connection with properties of their integral closure and to the 2-generator property [30, 31]. These rings are now called *finitely stable*.

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In a note of 1987, D.D. Anderson, J. Huckaba and I. Papick considered the notion of stability for integral domains [3]. If I is a nonzero ideal of a domain R , then the endomorphism ring of I coincides with the overring $E(I) = (I : I)$ of R ; also, I is projective over $E(I)$ if and only if I is invertible as an ideal of $E(I)$. We use here notations like $(I : I)$ in a more general context: If R and T are domains with the same field of fractions K , I is an ideal of R and S is a subset of K , we set $(I :_T S) = \{t \in T \mid tS \subseteq I\}$ and $(I : S) = (I :_K S)$. The stability property of a nonzero ideal I does not depend on the domain containing I : more precisely, if I is a common nonzero ideal of two domains A and B , then I is stable as an ideal of A if and only if I is stable as an ideal of B since $\text{Frac } A = \text{Frac } B$.

Since 1998, finitely stable and stable domains have been thoroughly investigated by Bruce Olberding in a series of papers [21]-[26]. In [27], he also studied finitely stable rings in the spirit of Rush, extending several results known for stable domains. Our paper heavily relies on Olberding's work. We thank B. Olberding for his valuable help. Also, as he communicated to us, his articles [22, 23, 24] contain some errors.

Of course, when R is a Noetherian ring, stability and finite stability coincide, but in general these two classes of rings are distinct, even if R is an integrally closed domain: in this case R is finitely stable if and only if it is Prüfer. Indeed, a domain R is integrally closed if and only if $R = E(I)$ for each nonzero finitely generated ideal I . However, a valuation domain is stable if and only if it is *strongly discrete*, that is, each nonzero prime ideal is not idempotent [5, Proposition 7.6]. Thus a valuation domain that is not strongly discrete is finitely stable, but not stable.

A domain R is finitely stable if and only if it is locally finitely stable [7, Proposition 7.3.4]. Actually, if I is a stable ideal of R , then I_S is a stable ideal of R_S for each multiplicative part $S \subseteq R$. A finitely stable domain need not have finite character, since any Prüfer domain is finitely stable. On the other hand, a domain is stable if and only if it is locally stable and has finite character [24, Theorem 3.3].

We denote by R' the integral closure of a domain R .

Olberding characterized finitely stable domains as follows:

Theorem 1.1. [27, Corollary 5.11] *A domain R is finitely stable if and only if it satisfies the following conditions:*

- (1) R' is a quadratic extension of R ;
- (2) R' is a Prüfer domain;
- (3) Each maximal ideal of R has at most 2 maximal ideals of R' lying over it.

Recall that a domain D is a *quadratic extension* of a domain R if for each $x, y \in D$ we have $xy \in xR + yR + R$. Olberding also proved that, in the local one-dimensional case, stability and finite stability are equivalent provided the maximal ideal is stable:

Proposition 1.2. [28, Theorem 4.1] *Let R be a local one-dimensional domain. The following conditions are equivalent:*

- (i) R is stable;
- (ii) R is finitely stable with stable maximal ideal;
- (iii) R' is a quadratic extension of R and R' is a Dedekind domain with at most two maximal ideals.

Stability is related to divisoriality and to the 2-generator property. Recall that an ideal I of a domain R is *divisorial* if $I \neq (0)$ and $I = I_v = (R : (R : I))$. A domain R is called *divisorial* if each nonzero ideal of R is divisorial, and it is called *totally divisorial* if each overring of R is divisorial. An ideal I of R is called *2-generated* if I can be generated by two elements. The domain R is *2-generated* if each finitely generated ideal of R is 2-generated.

A domain R is stable and divisorial if and only if it is totally divisorial [25, Theorem 3.12]. Also, any stable Noetherian domain is one-dimensional [33, Proposition 2.1], and a Noetherian domain is stable and divisorial (i.e., totally divisorial) if and only if it is 2-generated ([22, Theorem 3.1] and [5, Theorem 7.3]). The 2-generator property for Noetherian domains is strictly stronger than stability. The first example of a stable Noetherian domain that is not 2-generated (equivalently, it is not divisorial) was given in [33, Example 5.4]. Several other examples can be found in [25, Section 3].

A *Mori domain* is a domain with the ascending chain condition on divisorial ideals. This is equivalent to the property that each nonzero ideal I of R contains a finitely generated nonzero ideal J such that $(R : I) = (R : J)$, that is, $I_v = J_v$ [4, Theorem 2.1]. Clearly Noetherian domains are Mori. For the main properties of Mori domains, see the survey [4] and the references there. A nonzero ideal I of an integral domain R is *2- v -generated* if I contains a 2-generated ideal J such that $(R : I) = (R : J)$, and R is *2- v -generated* if each nonzero ideal of R is 2- v -generated. Of course, a 2- v -generated domain is Mori. However, if each divisorial ideal of R is principal (hence 2- v -generated), then R is not necessarily Mori (see [19, page 561]). Clearly, a Mori 2-generated domain is 2- v -generated.

A Mori domain R satisfies the ascending chain condition on principal ideals (for short, accp), and so it is *Archimedean*, that is, $\bigcap_{n \geq 0} r^n R = (0)$, for each nonunit $r \in R$. Indeed, a domain R satisfies accp if and only if $\bigcap_{n \geq 1} (\prod_{i=1}^n r_i R) = (0)$ for any nonunits $r_i \in R$, equivalently $\bigcap_{n \geq 1} a_n R = (0)$ if the sequence of principal ideals $a_n R$ is strictly decreasing. Besides accp domains, the class of Archimedean domains includes also one-dimensional domains [20, Corollary 1.4] and completely integrally closed domains [12, Corollary 13.4]. We recall that a domain R is completely integrally closed if and only if $R = E(I)$ for each nonzero ideal I . Hence completely integrally closed domains are integrally closed and the converse holds in the Noetherian case. A completely integrally closed stable domain is Dedekind.

As usual, if \mathcal{P} is a property of rings, then a ring R is *locally \mathcal{P}* if R_M is \mathcal{P} for each maximal ideal M of R . Generally, this does not imply that R_P is \mathcal{P} for every prime ideal P even for a local domain (see Example 6.8 for the Archimedean property). The property \mathcal{P} *localizes* if every ring satisfying \mathcal{P} is locally \mathcal{P} . The following properties localize: stability, finite stability, Mori. However, as it is well-known, the Archimedean property, the accp and the c.i.c. property do not localize (see Section 6 below).

Sections 2-5 contain our results. In Section 6 we present examples related mainly to the Archimedean property.

Here are our main results:

- (1) *A domain R is stable and one-dimensional if and only if it is finitely stable and Mori (Theorem 5.7). If R satisfies these two equivalent conditions, then each overring of R also satisfies these conditions and it is 2-v-generated.*
- (2) *A stable locally Archimedean domain satisfies accp (Corollary 3.20).*
- (3) *If R is an Archimedean finitely stable domain such that R' is equidimensional, then R is one-dimensional (Proposition 5.1).*
The assumption that R' is equidimensional is essential, as shown in Example 6.17.
- (4) *A stable radical ideal is divisorial (Corollary 2.21).*

A class of one-dimensional local domains that are stable and not Noetherian was constructed by Olberding in [26, Theorems 4.1 and 4.4] (see also [25, Theorem 3.10]). By our results, all these domains are new examples of one-dimensional Mori domains.

When studying the Archimedean property, we use Corollary 4.14: a stable domain R is Archimedean if and only if each nonunit of R belongs to a height-one prime ideal of R' (this result is related to Ohm's Theorem for Prüfer domains [20, Corollary 1.2]). We also prove that a stable domain is locally Archimedean if and only if $\bigcap_{n \geq 1} M^n = (0)$ for each maximal ideal M (Proposition 3.17); this condition implies accp (Proposition 3.19).

By Example 6.13, a stable Archimedean domain need not be locally Archimedean, and by Example 6.9 a semilocal Archimedean domain (even completely integrally closed) need not be locally Archimedean. On the positive side we show that an Archimedean finitely stable semilocal domain with stable maximal ideals is locally Archimedean (Proposition 4.16).

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2. THE ONE-DIMENSIONAL CASE

In the following, R is an integral domain that is not a field. By an *ideal* we mean an integral ideal.

The following construction, due to Olberding, is basic for our paper.

Construction 2.1. [24, Section 4] Let (R, M) be a local domain. Set $R_i = \{0\}$ for $i < 0$, $R_0 = R$ and $M_0 = M$. Define inductively for $n > 0$:

$R_n = R_{n-1}$ if R_{n-1} is not local, and $R_n = E(M_{n-1}) = (M_{n-1} : M_{n-1})$ if R_{n-1} is local with maximal ideal denoted by M_{n-1} . Set $T = \bigcup_{n \geq 0} R_n$.

Thus we have:

- (a) If there exists an integer $k > 0$ such that R_k is not local, but R_i is local for $0 \leq i < k$, then $R_n = R_k$ for all $n \geq k$, and $T = R_k$.
- (b) If $R_n \subsetneq R_{n+1}$ for all $n \geq 0$, all the rings R_n are local.

We will use repeatedly the following theorem of Olberding.

Theorem 2.2. [24, Corollary 4.3, Theorem 4.8] and its proof, and [27, Theorem 5.4] *Let R be a finitely stable local domain with stable maximal ideal M . With the notation of 2.1 we have:*

- (1) *Each R_n is finitely stable with stable maximal ideals, and there exists an element $m \in M$ such that $M = mR_1$. Moreover, for $k \geq 1$, if R_k is local with maximal ideal M_k , then $M_k = mR_{k+1} = MR_{k+1}$, and if T is local, then its maximal ideal is $mT = MT$.*
- (2) *Each R_n is a finitely generated R -module, thus T is an integral extension of R .*

We also have:

- (a) *If $T = R_n$ for some $n \geq 0$, then T is a finitely generated R -module, and T has at most two maximal ideals.*
- (b) *If $T \neq R_n$ for all $n \geq 0$, then T is local.*
- (c) *The maximal ideals of T are principal, and the Jacobson radical of T is equal to $mT = MT$, where $mR_1 = M$.*

In addition, if R is a stable domain, then T is equal to the integral closure R' of R , and R' is a strongly discrete Prüfer domain.

In the one-dimensional case we have:

Corollary 2.3. *Let R be a one-dimensional finitely stable local domain with stable maximal ideal M . Then R is stable, and in the setting of Theorem 2.2, $T = R'$ is a principal ideal domain with at most two maximal ideals. Hence, if T is local, in particular, if condition (b) holds, T is a DVR.*

Proof. R is stable by Proposition 1.2, so $T = R'$. Since R' is one-dimensional with principal maximal ideals, R' is a principal ideal domain by [12, Corollary 37.9]. \square

Proposition 2.4. *In the setting of Theorem 2.2, we have: condition (a) holds if and only if T is a finite R -extension (that is, T is a finitely generated R -module). Hence condition (b) holds if and only if T is not a finite R -extension. (Recall that if R is stable, then $T = R'$.)*

Proof. If condition (a) holds, then T is a finitely generated R -module by Theorem 2.2 (a). Conversely, assume that T is generated as an R -module by a finite subset F of T . Then there exists an integer $n \geq 0$ such that $F \subseteq R_n$, implying that $T = R_n$, so condition (a) holds. \square

In the Noetherian case, the next theorem was proved by Sally and Vasconcelos [32, Theorem 2.4]. Olberding proved that the hypotheses of the theorem imply Noetherianity.

Theorem 2.5. [23, Proposition 4.5] *Let R be a one-dimensional stable domain. If R' is a finite R -extension, then each ideal of R is 2-generated.*

In Proposition 2.11 below, we state that a stable one-dimensional local domain R is 2-*v-generated*, that is, for each nonzero ideal I there are two elements $x, y \in I$ such that $I_v = \langle x, y \rangle_v$; thus R is Mori.

Denote by $\mathcal{U}(A)$ the set of units of a domain A .

Remark 2.6. *In the setting of Theorem 2.2, for any integer $n \geq 0$ we have $\mathcal{U}(T) \cap R_n = \mathcal{U}(R_n)$, since T is an integral extension of R_n .*

Lemma 2.7. *Let R be a finitely stable local domain with stable maximal ideal. In the setting of Theorem 2.2, if T is local, in particular, if condition (b) holds, we have:*

- (1) *For each $n \geq 0$, $(R :_T m^n) = (R :_T M^n) = R_n$; equivalently, $Tm^n \cap R = R_n m^n$ (here $M^0 = R$).*
- (2) *Let $r = um^n$ be a nonzero element of R , where $u \in \mathcal{U}(T)$, and $n \geq 0$. Then $(R :_T r) = R_n$.*

Proof. (1) We prove the equality $(R :_T m^n) = R_n$ by induction on n starting with $n = 0$. Let $n > 0$. Since $M = R_1 m$, by applying the induction assumption to R_1 replacing R we obtain that:

$$(R :_T m^n) = (M :_T m^n) = (R_1 m :_T m^n) = (R_1 :_T m^{n-1}) = R_n.$$

Also $M^n = (R_1 m)^n = R_1 m^n$. Since $R_n = (R :_T m^n)$ and $R_1 \subseteq R_n$, we obtain

$$R_n \subseteq (R :_T R_1 m^n) = (R :_T M^n) \subseteq (R :_T m^n) = R_n,$$

so $(R :_T M^n) = R_n$.

(2) By item (1) we have $u \in R_n$, and also:

$$(R :_T r) = (R :_T um^n) = ((R :_T m^n) : u) = (R_n :_K u) = R_n,$$

where $K = \text{Frac } R$, since $u \in \mathcal{U}(R_n)$. □

Notation 2.8. In the setting of Theorem 2.2, assume that the domain R is one-dimensional and that T is local (in particular, T is local if condition (b) holds). As T is a DVR (Corollary 2.3) with maximal ideal mT , we denote by \mathbf{v} the discrete valuation of T such that $\mathbf{v}(m) = 1$.

Lemma 2.9. *In the setting of Theorem 2.2, assume that the domain R is one-dimensional and that T is local. Then, by using Notation 2.8, we have:*

- (1) *Let r be a nonzero element of R . Then:*

$$(R :_T r) = R_{\mathbf{v}(r)}.$$

- (2) Let I be a nonzero ideal of R , and let a be an element of minimal value $\mathbf{v}(a) = k$ in I . Then:

$$(R :_T I) = R_k.$$

Proof. (1) This follows from Lemma 2.7 (2).

(2) By item (1), we have

$$(R :_T I) = \bigcap_{r \in I \setminus \{0\}} (R :_T r) = \bigcap_{r \in I \setminus \{0\}} R_{\mathbf{v}(r)} = R_k.$$

□

From Lemma 2.7 (1) we obtain:

Lemma 2.10. *In the setting of Theorem 2.2, assume that the domain R is one-dimensional and that T is local. Then, in the notation 2.8, we have for all $k \geq 0$:*

$$\{r \in R \mid \mathbf{v}(r) \geq k\} = R \cap m^k T = R_k m^k.$$

Proposition 2.11. *A one-dimensional stable local domain R is 2- v -generated; hence R is a Mori domain.*

Proof. In case (a) of Theorem 2.2, every ideal of R is 2-generated by Theorem 2.5.

Assume condition (b) of Theorem 2.2, and use Notation 2.8. Let $I \neq R$ be a nonzero ideal of R . Since T is a DVR, there exists a nonzero element $t \in T$ of maximal value $\mathbf{v}(t)$ such that $\frac{1}{t}I \subseteq R$. Let $J = \frac{1}{t}I$, so $(R : J) \subseteq T$. Since $\frac{1}{m} \notin T$, there exists a nonzero element $a_1 \in J$ such that $\frac{a_1}{m} \notin R$. Let a_2 be an element of minimal value k in J . If $\frac{a_2}{m} \notin R$, set $a = a_2$. Assume that $\frac{a_2}{m} \in R$. If $\mathbf{v}(a_1) = \mathbf{v}(a_2)$, set $a = a_1$. Otherwise $\mathbf{v}(a_1) > \mathbf{v}(a_2)$, so $\mathbf{v}(a_1 + a_2) = \mathbf{v}(a_2)$ and $\frac{a_1 + a_2}{m} \notin R$. In this case we set $a = a_1 + a_2$. In each case, a is a nonzero element of minimal value k in J such that $\frac{a}{m} \notin R$. Thus $a = um^k$, where $u \in \mathcal{U}(R_k) \setminus R_{k-1}$, by Lemma 2.7 (1).

Since $(R : J) \subseteq T$ and $\frac{1}{um} \notin T$, there exists an element $b \in J$ such that $\frac{b}{um} \notin R$. We show that $(R : \{a, b\}) \subseteq T$.

If x is an element in $(R : \{a, b\}) \setminus T$, we have $x = \frac{1}{vm^i}$, where $v \in \mathcal{U}(T)$ and $i > 0$. Thus $\frac{1}{vm}a, \frac{1}{vm}b \in R$. Since $\frac{1}{vm}a = \frac{u}{v}m^{k-1} \in R$, we have $\frac{u}{v} \in \mathcal{U}(R_{k-1})$ by Lemma 2.7 (1). Since $\mathbf{v}(b) \geq k$, we have $\mathbf{v}(\frac{b}{vm}) \geq k-1$. As $\frac{b}{vm} \in R$, we obtain by Lemma 2.10 that $\frac{b}{vm} \in R_{k-1}m^{k-1}$. Hence, $\frac{b}{um} = \frac{v}{u}\frac{b}{vm} \in R_{k-1}m^{k-1} \subseteq R$, a contradiction. It follows that $(R : \{a, b\}) \subseteq T$.

Since a is of minimal value in J , by Lemma 2.9 (1)-(2), we have $(R :_T J) = (R :_T a)$.

Hence $(R : J) \subseteq (R : \{a, b\}) = (R :_T \{a, b\}) = (R :_T J) \subseteq (R : J)$, so $(R : J) = (R : \{a, b\})$. Thus J is 2- v -generated and so is $I = tJ$. We conclude that R is 2- v -generated. □

In Proposition 2.18 below we globalize Proposition 2.11.

Lemma 2.12. *Let S be a multiplicative subset of an integral domain R . If I is a 2- v -generated nonzero ideal of R , then the ideal IR_S of R_S is 2- v -generated. Hence, if R is 2- v -generated, also R_S is 2- v -generated.*

Proof. There exists a 2-generated subideal J of I such that $(R : J) = (R : I)$. Since $(R : J)R_S = (R_S : JR_S)$, we have $(R_S : IR_S) = (R_S : JR_S)$ and so the ideal IR_S of R_S is 2- v -generated. \square

Lemma 2.13. *Let (R, M) be a local one-dimensional domain, and let $a, b \in M$ be two nonzero elements. Then each element in $a + Rb^k$ is associated with a for all sufficiently large integers k .*

Proof. Since R is local and one-dimensional, we have $M = \sqrt{aM}$, so $b^k \in aM$ for each sufficiently large integer k . Hence for all $r \in R$ we have $a + rb^k = a(1 + r(\frac{b^k}{a}))$, where $1 + r(\frac{b^k}{a})$ is a unit in R since $\frac{b^k}{a} \in M$. \square

Proposition 2.14. *Let R be a one-dimensional domain of finite character. The following conditions are equivalent:*

- (i) R is 2- v -generated;
- (ii) R is locally 2- v -generated.

Proof. (i) \Rightarrow (ii) If R is 2- v -generated, then R is locally 2- v -generated by Lemma 2.12.

(ii) \Rightarrow (i) Assume that R is locally 2- v -generated. We prove that each nonzero ideal $I \neq R$ of R is 2- v -generated. Since R has finite character there are just finitely many maximal ideals containing I , say M_1, \dots, M_e , which we assume to be distinct. For each $1 \leq i \leq e$, the domain R_{M_i} is 2- v -generated, so there exist nonzero elements a_i, b_i in I such that $(R_{M_i} : I) = (R_{M_i} : \{a_i, b_i\})$. There exist pairwise comaximal elements $m_i \in M_i$, for $1 \leq i \leq e$. By the Chinese Remainder Theorem, for each positive integer k there exists an element $a \in I$ such that we have in R :

$$a \equiv a_i \pmod{Im_i^k}$$

for $1 \leq i \leq e$. By Lemma 2.13, we may choose k sufficiently large such that for each i the elements a and a_i are associated in R_{M_i} , so $(R_{M_i} : I) = (R_{M_i} : \{a_i, b_i\}) = (R_{M_i} : \{a, b_i\})$.

Let N_q ($q = 1, 2, \dots, f$) be the maximal ideals containing a but not I . There exist pairwise comaximal elements $n_i \in M_i$, for $1 \leq i \leq e$ that belong to no maximal ideal N_q . Also there exists an element $c \in I$ that belongs to no ideal N_q . By the Chinese Remainder theorem, for each positive integer j there exists an element $b \in I$ such that $b \equiv b_i \pmod{In_i^j}$ for each $1 \leq i \leq e$, and $b \equiv c \pmod{IN_q}$ for each ideal N_q . Hence $b \notin N_q$ for all $1 \leq q \leq f$. By Lemma 2.13, for a sufficiently large integer j and for each i , the elements b and b_i are associated in R_{M_i} , so $(R_{M_i} : I) = (R_{M_i} : \{a, b\})$ for all $1 \leq i \leq e$.

Let M be a maximal ideal of R . If M contains I , thus $M = M_i$ for some integer $1 \leq i \leq e$, then $(R_{M_i} : I) = (R_{M_i} : \{a, b\})$. If M contains a but not I , then $b \equiv 1 \pmod{M}$, so $b \notin M$. Thus $(R_M : I) = R_M = (R_M : \{a, b\})$.

If M does not contain a , then again $(R_M : I) = R_M = (R_M : \{a, b\})$. Thus for each maximal ideal M of R we have $(R_M : I) = (R_M : \{a, b\})$. Hence

$$(R : I) = \bigcap_M (R_M : I) = \bigcap_M (R_M : \{a, b\}) = (R : \{a, b\}),$$

where M runs over all the maximal ideals of R . We conclude that I is 2- v -generated, so the domain R is 2- v -generated. \square

A locally 2- v -generated domain R is not necessarily 2- v -generated even if R is one-dimensional. For example, if R is an almost Dedekind domain that is not Dedekind, then R is locally a DVR, but R is not Mori since an almost Dedekind and Mori domain is Dedekind. For a positive result, see Proposition 2.17 below.

Corollary 2.15. *A one-dimensional stable domain is 2- v -generated if and only if it is locally 2- v -generated*

Proof. Indeed, a stable domain has finite character. \square

Lemma 2.16. *A one-dimensional Mori domain has finite character.*

Proof. If R is Mori and one-dimensional, every maximal ideal of R is divisorial [4, Theorem 3.1]. By [4, Theorem 3.3 (c)], a Mori domain is an intersection of finite character of the localizations at its maximal divisorial ideals. It follows that R has finite character. \square

Proposition 2.17. *Let R be a one-dimensional domain. The following conditions are equivalent:*

- (i) R is 2- v -generated;
- (ii) R is locally 2- v -generated and R has finite character.

Proof. (i) \Rightarrow (ii) R is locally 2- v -generated by Lemma 2.12 and has finite character by Lemma 2.16.

(ii) \Rightarrow (i) See Proposition 2.14. \square

Proposition 2.18. *A one-dimensional stable domain R is 2- v -generated; hence R is Mori.*

Proof. Since R is locally stable, R is locally 2- v -generated by Proposition 2.11. Thus R is 2- v -generated by Corollary 2.15. \square

The stability assumption in Propositions 2.11 and 2.18 cannot be relaxed to finite stability. Indeed, let R be a one-dimensional valuation domain that is not a DVR. Thus R is finitely stable, but R is neither Mori, nor stable (the maximal ideal of R is not stable); see [23, Example 3.3]. On the other hand, we prove below that a one-dimensional finitely stable Mori domain is stable (Proposition 2.22). To prove Proposition 2.22 we need a few preliminary results.

Proposition 2.19. *Let I be a stable ideal of an integral domain R . Then $I_v = I(I_v : I_v)$ is stable, and $(I_v)^2 \subseteq I$.*

Proof. Let $D = (I_v : I_v)$. Thus $(I : I) \subseteq (I_v : I) = (I_v : I_v) = D$. Since I is an invertible ideal of $(I : I)$ and $(I : I) \subseteq D$, it follows that ID is an invertible ideal of D . As D is a fractional divisorial ideal of R , we obtain that ID is a fractional divisorial ideal of R . Hence $I_v \subseteq ID$, so $I_v = ID$ since I_v is an ideal of D . Thus $I_v = ID$ is invertible in $D = (I_v : I_v)$, that is, I_v is a stable ideal of R . Also $(I_v)^2 = I_v(ID) = (I_v D)I = I_v I \subseteq I$. \square

Corollary 2.20. [10, Lemma 2.7] *In a finitely stable domain all the v -finite divisorial ideals are stable. In particular, all the divisorial ideals of a finitely stable Mori domain are stable.*

A nonzero ideal I of a domain is called a t -ideal if $I = \bigcup J_v$, where J runs over all finitely generated subideals of I . Divisorial ideals are t -ideals, and in a Mori domain each t -ideal is divisorial.

Corollary 2.21.

- (1) *A stable radical ideal is divisorial.*
(Cf. [24, Corollary 4.13]. Here we do not assume that the domain R is stable).
- (2) *If I is a radical ideal and each finitely generated subideal of I is stable, then I is a t -ideal.*
- (3) *Each nonzero radical ideal of a finitely stable domain is a t -ideal.*
- (4) *All the nonzero radical ideals of a finitely stable Mori domain are divisorial and stable.*

Proof. (1) Let I be a stable radical ideal of R . By Proposition 2.19, we have $(I_v)^2 \subseteq I$, so $I_v \subseteq I$ as the ideal I is radical. Hence $I = I_v$ is a divisorial ideal.

(2) If J is a nonzero finitely generated subideal of I , then $(J_v)^2 \subseteq J \subseteq I$ by Proposition 2.19. Since the ideal I is radical, we obtain $J_v \subseteq I$, so I is a t -ideal.

(3) follows from (2).

(4) All the radical ideals of a Mori domain are divisorial by item (2), so they are also stable by Corollary 2.20. \square

Proposition 2.22. *A one-dimensional finitely stable Mori domain is stable.*

Proof. For each maximal ideal M of R , R_M is Mori and finitely stable. Hence MR_M is divisorial (Corollary 2.21 (3)) and so stable (Corollary 2.20). By Proposition 1.2, R_M is stable. Since R has finite character (Lemma 2.16), R is stable by [24, Theorem 3.3]. \square

Actually, as shown in Theorem 5.7 below, a finitely stable Mori domain is one-dimensional, so it is stable and 2- v -generated (Proposition 2.22 and Proposition 2.18).

3. ON THE ARCHIMEDEAN PROPERTY

We start with some generalities on the Archimedean property. Then we prove that a finitely stable domain R with stable maximal ideals is locally Archimedean if and only if $\bigcap_{n \geq 1} M^n = (0)$ for each maximal ideal M of R (Proposition 3.17). We deduce from this result that a locally Archimedean stable domain satisfies accp (Corollary 3.20).

Many results in this section are related to the following theorem of J. Ohm, which will be extended in Theorem 4.13 below.

Theorem 3.1. [20, Corollary 1.6]. *Let R be a Prüfer domain. We have:*

(1) *If a is a nonunit of R belonging to just finitely many maximal ideals, then $\bigcap_{n \geq 1} a^n R = (0)$ if and only if a belongs to a height-one prime ideal.*

Hence:

(2) *If R has finite character, then R is Archimedean if and only if each nonunit of R belongs to a height-one prime ideal.*

Corollary 3.2. *An Archimedean Prüfer domain of finite character and with just finitely many height-one prime ideals is one-dimensional. In particular, an Archimedean Prüfer semilocal domain is one-dimensional.*

Proof. Let M be a maximal ideal of R . By Ohm's Theorem 3.1 (2), M is contained in the finite union of the height-one prime ideals of R . Hence M has height one, so R is one-dimensional.

If R is Prüfer and semilocal, then R has just finitely many height-one prime ideals. Hence, if R is Archimedean, then R is one-dimensional. \square

Remark 3.3. *An integral domain R is Archimedean if and only if for each nonzero nonunit r of R there is an Archimedean domain D (depending on r) containing R such that r is a nonunit in D . Moreover, replacing D by $D \cap \text{Frac}(R)$, we may assume that D is an overring of R .*

In particular, an intersection of Archimedean domains is Archimedean. Hence a locally Archimedean domain is Archimedean.

Corollary 3.4. *A domain R is Archimedean if and only if R has an Archimedean integral extension overring.*

Corollary 3.5. *Let $A \subseteq B$ be an extension of integral domains. If every nonzero nonunit of A belongs to a height-one prime ideal of B , then A is Archimedean.*

Proof. Let a be a nonzero nonunit of A . If Q is a height-one prime ideal of B containing a , then a is a nonunit in the one-dimensional (so Archimedean) domain B_Q . By Remark 3.3, A is Archimedean. \square

Corollary 3.6. *Let (R, M) be a local domain. If some integral extension of R has a height-one maximal ideal, then R is Archimedean.*

Proof. If Q is a height-one maximal ideal of an integral extension D of R , then $Q \cap R = M$. Hence R is Archimedean by Corollary 3.5. \square

Proposition 3.7. *Let R be an integral domain, and let a be a nonunit of R that belongs to just finitely many maximal ideals.*

Then $\bigcap_{n \geq 1} a^n R = (0)$ if and only if a belongs to a maximal ideal M such that $\bigcap_{n \geq 1} a^n R_M = (0)$.

Proof. Let \mathfrak{F} be the set of maximal ideals containing a . We have

$$\bigcap_{n \geq 1} a^n R = R \cap \bigcap_{n \geq 1} \left(\bigcap_{M \in \mathfrak{F}} a^n R_M \right) = R \cap \bigcap_{M \in \mathfrak{F}} \left(\bigcap_{n \geq 1} a^n R_M \right) = \bigcap_{M \in \mathfrak{F}} R \cap \left(\bigcap_{n \geq 1} a^n R_M \right).$$

Since the set \mathfrak{F} is finite it follows that $\bigcap_{n \geq 1} a^n R = (0)$ if and only if $R \cap \bigcap_{n \geq 1} (a^n R_M) = (0)$ for some $M \in \mathfrak{F}$, equivalently $\bigcap_{n \geq 1} a^n R_M = (0)$ for some $M \in \mathfrak{F}$. \square

We will often use the following well-known fact:

Lemma 3.8. [12, Theorem 7.6 (a) and (c)] *Let P be an invertible prime ideal of an integral domain R . Then $\bigcap_{n \geq 1} P^n$ is the largest prime ideal properly contained in P .*

Proposition 3.9. *If R is an Archimedean domain and P is a principal prime ideal of R , then R_P is a DVR.*

Proof. If $P = rR$, then by Lemma 3.8 $\bigcap_{n \geq 0} P^n = \bigcap_{n \geq 0} r^n R = (0)$ is the largest prime ideal of R properly contained in P . It follows that R_P is a one-dimensional local domain with principal maximal ideal, and so R_P is a DVR. \square

Corollary 3.10. *Let R be an integral domain.*

- (1) *If R is Archimedean with principal maximal ideals, then R is a principal ideal domain.*
- (2) *If R is locally Archimedean with invertible maximal ideals, then R is a Dedekind domain.*

Proof. (1) By Proposition 3.9, R is one-dimensional. Since every nonzero prime ideal of R is principal, R is a principal ideal domain by [12, Corollary 37.9].

(2) By Proposition 3.9, R is locally a DVR (i.e., R is almost Dedekind); in particular R is one-dimensional. It follows that R is a Dedekind domain by [12, Theorem 37.8 (1) \Leftrightarrow (4)]. \square

However, an Archimedean domain R with invertible maximal ideals is not necessarily one-dimensional, even if R is Prüfer and stable: see Example 6.13 below.

Corollary 3.11. *An Archimedean Bézout domain R with stable maximal ideals is a principal ideal domain.*

Proof. As mentioned at the end of the proof of [24, Lemma 4.5], a stable maximal ideal M of a Prüfer domain R is invertible since $(M : M) = R$. Thus the maximal ideals of R are finitely generated, so they are principal. Hence R is a principal ideal domain by Corollary 3.10. \square

None of the two conditions on the Bézout domain R in Corollary 3.11 to be a principal ideal domain can be omitted. Indeed, $R = \mathbb{Z} + X\mathbb{Q}[X]$ is a two-dimensional Bézout domain with principal maximal ideals. On the other hand, the ring of entire functions is an infinite-dimensional completely integrally closed Bézout domain. Thus R is Archimedean; see also Remark 3.12 below. Hence R has non-principal maximal ideals: these are the free maximal ideals: see [7, Ch. VIII, §8.1] and [29, Ch.6, §3].

Remark 3.12. *By [34, Corollary 2.4], a GCD domain (in particular, a Bézout) domain is Archimedean if and only if it is completely integrally closed.*

Lemma 3.13. *Let I and J be two ideals of a ring R . If I contains a power of J , then*

$$\bigcap_{n \geq 1} J^n \subseteq \bigcap_{n \geq 1} I^n.$$

Hence, if $J \subseteq \sqrt{I}$ and the ideal J is finitely generated, then

$$\bigcap_{n \geq 1} J^n \subseteq \bigcap_{n \geq 1} I^n.$$

Proof. Let $J^k \subseteq I$ for some $k \geq 1$. Then $\bigcap_{n \geq 1} J^n = \bigcap_{n \geq 1} (J^k)^n \subseteq \bigcap_{n \geq 1} I^n$. If $J \subseteq \sqrt{I}$ is finitely generated, then I contains a power of J . \square

Corollary 3.14. *Let I be an ideal of an integral domain R . If $\bigcap_{n \geq 1} a^n R = (0)$ for all $a \in I$, then $\bigcap_{n \geq 1} a^n R = (0)$ for all $a \in \sqrt{I}$.*

Lemma 3.15. [27, Corollary 5.7] *Let R be a finitely stable local domain. Then a stable ideal I of R is principal in $(I : I)$. Moreover, if $I = x(I : I)$, then $I^2 = xI$.*

Lemma 3.16. *Let R be a finitely stable local domain with stable maximal ideal M . Then M is the radical of a principal ideal and*

$$\bigcap_{n \geq 0} M^n = \bigcap_{n \geq 0} a^n R$$

for each element $a \in R$ such that $\sqrt{aR} = M$.

Proof. By Lemma 3.15, there exists an element $m \in M$ such that $M^2 = mM$. Clearly $\bigcap_{n \geq 0} M^n = \bigcap_{n \geq 0} m^n R$, and $M = \sqrt{mR}$. If $\sqrt{aR} = M$, then $\sqrt{aR} = \sqrt{mR}$, so

$$\bigcap_{n \geq 0} a^n R = \bigcap_{n \geq 0} m^n R = \bigcap_{n \geq 0} M^n,$$

by Lemma 3.13. \square

Proposition 3.17. *Let R be a finitely stable domain with stable maximal ideals. Then R is locally Archimedean if and only if $\bigcap_{n \geq 1} M^n = (0)$ for each maximal ideal M .*

Proof. By Lemma 3.16, R is locally Archimedean if and only if $\bigcap_{n \geq 1} M^n R_M = (0)$ for every maximal ideal M . On the other hand, for every maximal ideal M we have

$$\bigcap_{n \geq 1} M^n = \bigcap_{n \geq 1} (M^n R_M \cap R) = \left(\bigcap_{n \geq 1} M^n R_M \right) \cap R,$$

so R_M is Archimedean if and only if $\bigcap_{n \geq 1} M^n = (0)$. The proposition follows. \square

Remark 3.18. *For any integral domain R , the following two conditions are equivalent:*

- (i) $\bigcap_{n \geq 1} I^n = (0)$ for each ideal I ;
- (ii) $\bigcap_{n \geq 1} M^n = (0)$ for each maximal ideal M .

If R satisfies these conditions, then R is locally Archimedean by Proposition 3.17.

Proposition 3.19. *Let R be an integral domain of finite character such that $\bigcap_{n \geq 1} M^n = (0)$ for each maximal ideal M of R . Then R satisfies accp.*

Proof. Assume that R does not satisfy accp. Then there exists an infinite sequence of nonunits r_n in R such that $\bigcap_{n \geq 1} (\prod_{i=1}^n r_i R) \neq (0)$. Let c be an element in this intersection. For all $n \geq 1$, each maximal ideal containing r_n contains also c , since $c \in r_n R$. As c belongs to just finitely many maximal ideals, there exists a maximal ideal M containing c such that $r_n \in M$ for infinitely many n 's. Thus for each $n \geq 1$, there exist integers $1 \leq i_1 < i_2 < \dots < i_n$ such that $r_{i_k} \in M$ for all $1 \leq k \leq n$. We have $c \in \prod_{j=1}^{i_n} r_j R \subseteq M^n$. Hence $c \in \bigcap_{n \geq 1} M^n$, a contradiction. \square

From Proposition 3.19 we obtain, by using Proposition 3.17:

Corollary 3.20. *A locally Archimedean finitely stable domain with stable maximal ideals and of finite character, in particular, a locally Archimedean stable domain, satisfies accp.*

However a domain R of finite character satisfying accp is not necessarily locally Archimedean, even if R is stable (see Example 6.13 below).

4. AN EXTENSION OF OHM'S THEOREM TO FINITELY STABLE DOMAINS

By using Ohm's Theorem 3.1 and the fact that an integral extension overring of a finitely stable domain is quadratic (Theorem 1.1), so algebraically bounded, as defined in 4.1 below, we extend Ohm's Theorem from Prüfer domains to finitely stable domains (Theorem 4.13). We present a criterion for the locally Archimedean property of a stable domain in Proposition 4.15. As an application, we prove that a semilocal finitely stable Archimedean domain is locally Archimedean (Proposition 4.16).

Definition 4.1. Let $A \subseteq B$ be an extension of integral domains. The domain B is a *bounded algebraic extension* of A if there exist a nonzero element $d \in A$ and an integer $e \geq 1$ such that for each element $b \in B$ there exists a monic polynomial $f(X)$ of degree e in $A[X]$ satisfying $f(db) = 0$. The domain B is called a *bounded integral extension* of A if this property holds for $d = 1$.

Remark 4.2. Let $A \subseteq B$ be an extension of integral domains. Then:

- (1) B is a bounded algebraic extension of A if and only if there exists a nonzero element $d \in B$ such that $A + dB$ is a bounded integral extension of A .
- (2) If $(A : B) \neq (0)$, then B is a bounded algebraic extension of A .

Proposition 4.3. Let A be an integral domain, let B be a bounded algebraic overring of A , and let a be an element of A . Then

$$\bigcap_{n \geq 1} a^n A = (0) \Leftrightarrow \bigcap_{n \geq 1} a^n B = (0).$$

Hence, if B is Archimedean, also A is Archimedean.

Proof. Assume that $\bigcap_{n \geq 1} a^n A = (0)$. Let b be an element in $\bigcap_{n \geq 1} a^n B$. Since B is a bounded algebraic extension of A , there exist a nonzero element $d \in A$ and an integer $e \geq 1$ such that for each $x \in B$, the element dx is a root of a monic polynomial of degree e in $A[X]$. Thus, for each $b \in B$ and $n \geq 1$, by taking $x = \frac{b}{a^n}$, there exist elements $a_0, \dots, a_{e-1} \in A$ (depending on b and on n) such that we have:

$$(1) \quad \left(\frac{db}{a^n}\right)^e + a_{e-1} \left(\frac{db}{a^n}\right)^{e-1} + \dots + a_0 = 0.$$

Since B is an overring of A , there exists a nonzero element $c \in A$ (depending just on b) such that $c(db)^i \in A$ for all $1 \leq i \leq e$. Multiplying the equation (1) by $ca^{n(e-1)}$ we obtain that $\frac{c(db)^e}{a^n} \in A$ for all $n \geq 1$. Hence $b = 0$. We conclude that $\bigcap_{n \geq 1} a^n B = (0)$. The proposition follows. \square

Corollary 4.4. Let A be a finitely stable domain, let a be a nonzero element of A , and let B be an integral extension overring of A . Then:

$$\bigcap_{n \geq 1} a^n A = (0) \Leftrightarrow \bigcap_{n \geq 1} a^n B = (0).$$

Proof. By Theorem 1.1, B is a quadratic extension of A , so B is a bounded integral extension of A . The corollary follows from Proposition 4.3. \square

Proposition 4.5. Let (R, M) be a finitely stable local domain with stable maximal ideal, and let D be an integral extension overring of R .

Then R is Archimedean if and only if D has a maximal ideal N such that D_N is Archimedean.

Proof. Assume that R is Archimedean. Let $M = m(M : M)$, $m \in M$ (Lemma 3.15 or Theorem 2.2). By Corollary 4.4, $\bigcap_{n \geq 1} m^n D = (0)$. By Theorem 1.1, D has at most two maximal ideals. By Proposition 3.7, there exists a maximal ideal N of D such that $\bigcap_{n \geq 1} m^n D_N = (0)$. Since $M^2 = mM$, we see that $\bigcap_{n \geq 1} M^n D_N = (0)$. Since \bar{D} is an integral extension of R and R is local, it follows that a prime ideal of D contains M if and only if it is a maximal ideal of D . Hence the only prime ideal of D_N containing MD_N is ND_N , so $ND_N = \sqrt{MD_N}$. By Corollary 3.14, D_N is Archimedean.

Conversely, if D_N is Archimedean, then R is Archimedean by Remark 3.3 since $R \subseteq D_N$ and $N \cap R = M$. \square

Corollary 4.6. *Let (R, M) be a finitely stable local domain with stable maximal ideal, and let D be an integral extension overring of R . Assume that if N is a maximal ideal of D such that the domain D_N is Archimedean, then D_N is one-dimensional. Then R is Archimedean if and only if D has a height-one maximal ideal.*

Proposition 4.7. *Let (R, M) be a local domain.*

- (1) *If some integral extension of R has a height-one maximal ideal, then R is Archimedean.*
- (2) *Conversely, we have:*
 - (a) *If R is Archimedean and finitely stable, then R' has a height-one maximal ideal.*
 - (b) *If R is Archimedean, finitely stable and the ideal M is stable, then T has a height-one maximal ideal (T is defined in Construction 2.1).*

Proof. (1) is Corollary 3.6.

(2, a) By Theorem 1.1, R' has at most two maximal ideals. Since R' is Prüfer, R' has at most two height-one prime ideals: Q_1 and Q_2 (not necessarily distinct). Let $P_i = Q_i \cap R$, $i = 1, 2$. Since R is Archimedean, by Corollary 4.4 we have $\bigcap_{n \geq 1} a^n R' = (0)$ for all $a \in R$. By Theorem 3.1, $M \subseteq P_1 \cup P_2$. We may assume that $M \subseteq P_1$, so $M = P_1 = Q_1 \cap R$. Hence Q_1 is a height-one maximal ideal of R' .

(2, b) By Proposition 4.5, T has a maximal ideal N such that the domain T_N is Archimedean. Hence T_N is a DVR by Proposition 3.9 as N is a principal ideal. Thus N is a height-one maximal ideal of T . \square

In the notation of 2.1, if R_k is one-dimensional for some $k \geq 0$, then all the rings R_n , as well as T , are one-dimensional since T is an integral extension of R_n , for all $n \geq 0$. For the Archimedean property we have:

Corollary 4.8. *Let (R, M) be a finitely stable local domain with stable maximal ideal. Set $R_\infty = T = \bigcup_{n \geq 0} R_n$ (see Construction 2.1). Assume that R_k is Archimedean for some $0 \leq k \leq \infty$. Then R_n is Archimedean for each n such that R_n is local. Thus R_n is Archimedean at least for each $R_n \neq T$.*

Proof. For all $0 \leq n \leq \infty$ we have $(R_n)' = R'$, so the corollary follows from Proposition 4.7. \square

Corollary 4.8 might fail when T is not local, so $T = R_n$ for some integer n . Indeed, in Example 6.17, R is Archimedean, but $T = R' = R_1$ is not Archimedean. Moreover, we have:

Proposition 4.9. *Let R be a finitely stable local domain with stable maximal ideal. In the notation of 2.1, T is Archimedean if and only if R is one-dimensional.*

Proof. If R is one-dimensional, then T is one-dimensional, and so Archimedean, since T is an integral extension of R . Conversely, if T is Archimedean, then T , and so also R , is one-dimensional by Corollary 3.10, as the maximal ideals of T are principal \square

From Proposition 4.7 (2, b) we obtain:

Corollary 4.10. *Let R be an Archimedean finitely stable local domain with stable maximal ideal. In the notation of 2.1, if T is local, in particular, if condition (b) of Theorem 2.2 holds, then R is one-dimensional.*

We now present an alternative proof of Corollary 4.10 (Proposition 4.12). We use the following lemma:

Lemma 4.11. *Let (R, M) be a finitely stable local domain with stable maximal ideal. In the notation of 2.1 assume that T is local. Then*

$$\left(\bigcap_{n \geq 0} m^n T \right)^2 \subseteq \bigcap_{n \geq 0} m^n R.$$

Proof. By Lemma 2.7 (1), we have for all $n \geq 0$:

$$\left(R \cap \bigcap_{k \geq 0} m^k T \right)^2 \subseteq (R \cap m^n T)^2 = (m^n R_n)^2 = m^n (m^n R_n) \subseteq m^n R,$$

so $\left(R \cap \bigcap_{k \geq 0} m^k T \right)^2 \subseteq \bigcap_{n \geq 0} m^n R$.

Now let $s, t \in \bigcap_{n \geq 0} m^n T$. Again by Lemma 2.7 (1), we have $sm^e, tm^e \in R$ for a sufficiently large integer e . Thus $(sm^e)(tm^e) \in \left(R \cap \bigcap_{n \geq 0} m^n T \right)^2 \subseteq \bigcap_{n \geq 0} m^n R$. It follows that $st = \frac{(sm^e)(tm^e)}{m^{2e}} \in \bigcap_{n \geq 0} m^n R$. Hence $\left(\bigcap_{n \geq 0} m^n T \right)^2 \subseteq \bigcap_{n \geq 0} m^n R$. \square

Proposition 4.12 (Corollary 4.10). *Let (R, M) be a finitely stable Archimedean local domain with stable maximal ideal, and such that T is local (in the notation of 2.1). Then R is one-dimensional.*

Proof. By Theorem 2.2, the maximal ideal of T is mT , $m \in M$. Let $Q = \bigcap_{n \geq 0} m^n T$. By Lemma 4.11, $Q^2 \subseteq \bigcap_{n \geq 0} m^n R = (0)$. Hence $Q = (0)$. By Proposition 3.8, Q is the largest non-maximal prime contained in mT . Thus T is one-dimensional, and so is R , as T is an integral extension of R . \square

We now state the promised generalization of Ohm's Theorem 3.1.

Theorem 4.13. *Let R be a finitely stable domain, and let a be a nonzero nonunit of R belonging to just finitely many maximal ideals of R . The following conditions are equivalent:*

- (i) $\bigcap_{n \geq 1} a^n R = (0)$;
- (ii) a belongs to a height-one prime ideal of R' ;
- (iii) a belongs to a prime ideal P of R such that the domain R_P is Archimedean.

Proof. (i) \Rightarrow (ii) By Corollary 4.4, $\bigcap_{n \geq 1} a^n R' = (0)$. If N is a maximal ideal of R' containing a , then $N \cap R$ is a maximal ideal of R containing a . Since each maximal ideal of R is contained in at most two maximal ideals of R' (Theorem 1.1), it follows that a belongs to just finitely many maximal ideals of R' . Since R' is Prüfer, by Theorem 3.1, a belongs to a height-one prime ideal of R' .

(ii) \Rightarrow (iii) Let Q be a height-one prime ideal of R' containing a , and let $P = Q \cap R$. By Corollary 3.5 for $A = R_P$ and $B = R'_Q$, we obtain that R_P is Archimedean.

(iii) \Rightarrow (i) follows from Remark 3.3. □

Corollary 4.14. *Let R be a finitely stable domain of finite character (this holds, in particular, if R is a stable domain). Then R is Archimedean if and only if every nonzero nonunit in R belongs to a height-one prime ideal of R' .*

Proposition 4.15. *Let R be a finitely stable domain. The following conditions are equivalent:*

- (i) R is locally Archimedean;
- (ii) Each maximal ideal of R is contained in a height-one prime ideal of R' (which is necessarily maximal);
- (iii) Each proper ideal of R is contained in a height-one maximal ideal of R' .

Proof. (i) \Rightarrow (ii) If R is local, then (ii) follows from Proposition 4.7(2)(a).

In the general case, let M be a maximal ideal of R . By the local case, the ideal MR_M of R_M is contained in a height-one prime Q of $(R_M)' = R'_M$, where R'_M is the localization of R' at the multiplicative subset $R \setminus M$. Thus $Q \cap R'$ is a height-one prime ideal of R' containing M .

(ii) \Rightarrow (i) Let M be a maximal ideal of R . Let Q be a height-one prime ideal of R' containing M . Thus QR'_M is a height-one prime ideal of $R'_M = (R_M)'$ containing M . By Corollary 3.6, R_M is Archimedean, so R is locally Archimedean.

(ii) \Leftrightarrow (iii) Clear. □

In the next proposition we extend Corollary 3.2 to finitely stable domains:

Proposition 4.16. *An Archimedean finitely stable domain of finite character such that its integral closure has just finitely many height-one prime ideals is locally Archimedean. In particular, an Archimedean finitely stable semilocal domain is locally Archimedean.*

Proof. Let M be a maximal ideal of R . As R is Archimedean, by Theorem 4.13, M is contained in the finite union of the height-one primes of R' . Thus the ideal MR' of R' is contained in one of these primes. By Proposition 4.15, R is locally Archimedean.

If R is an Archimedean finitely stable semilocal domain, then R' is Prüfer and semilocal. Thus R' has just finitely many height-one prime ideals. It follows that R is locally Archimedean. \square

In connection with Proposition 4.16, by Example 6.13, a stable Archimedean domain need not be locally Archimedean, and by Example 6.9 a semilocal Archimedean (even completely integrally closed) domain need not be locally Archimedean.

Question 4.17. *By Proposition 4.15, if a finitely stable domain R is locally Archimedean, then each nonzero nonunit of R belongs to a height-one maximal ideal of R' . Is the converse true? Cf. Corollary 4.14.*

5. ONE-DIMENSIONALITY OF ARCHIMEDEAN STABLE DOMAINS

In this section, we prove that a finitely stable Mori domain is one-dimensional, so it is stable and 2- v -generated (Theorem 5.7). Thus we get rid of the one-dimensional assumption in Proposition 2.22.

We also illustrate a general method for constructing a local n -dimensional Archimedean stable domain for each integer n (Propositions 5.12 and 5.13); see also Example 6.17 below.

First we state a criterion for one-dimensionality of an Archimedean stable domain. We say that a domain R is *equidimensional* if $\dim R = \dim R_M$, for each maximal ideal M .

Proposition 5.1. *Let R be an Archimedean finitely stable domain of finite character (this includes the case that R is Archimedean and stable). The following conditions are equivalent:*

- (i) R is one-dimensional;
- (ii) Every integral extension of R is equidimensional;
- (iii) R' is equidimensional;
- (iv) The pair (R, R') satisfies GD (the going down property) and R is equidimensional.

Proof. (i) \Rightarrow (ii) Every integral extension of R is one-dimensional, so also equidimensional.

(ii) \Rightarrow (iii) Obvious.

(iii) \Rightarrow (i) By Corollary 4.14, R' has a height-one maximal ideal. Thus R' is one-dimensional, and so is R .

(i) \Rightarrow (iv) Clear.

(iv) \Rightarrow (iii) Indeed, if B is any ring extension of an equidimensional (in particular, local) ring A such that the pair (A, B) satisfies GD, then B is equidimensional and $\dim B = \dim A$. \square

Proposition 5.2. *Let R be an Archimedean finitely stable local domain with stable maximal ideal. The following conditions are equivalent:*

- (i) R is one-dimensional;
- (ii) T is Archimedean;
- (iii) T is equidimensional;
- (iv) The pair (R, T) satisfies GD.

(See 2.1 for the notation T .)

Proof. (i) \Leftrightarrow (ii) is Proposition 4.9.

(i) \Rightarrow (iii) because T is one-dimensional.

(iii) \Rightarrow (i) T has a height-one maximal ideal by Proposition 4.7 (2,b). Thus T is one-dimensional, and so is R .

(i) \Rightarrow (iv) This follows from that both R and T are one-dimensional.

(iv) \Rightarrow (iii) Since R is local, we may use the proof of the implication (iv) \Rightarrow (iii) in Theorem 5.1. \square

Proposition 5.3. *Let R be an Archimedean finitely stable semilocal domain. Then R is one-dimensional if and only if the pair (R, R') satisfies GD.*

Proof. Assume that the pair (R, R') satisfies GD. Let M be a maximal ideal of R . By Corollary 4.14, M is contained in the union of the height-one maximal ideals of R' . Since R' is semilocal (Theorem 1.1), it follows that M is contained in a height-one maximal ideal N of R' . As the pair (R, R') satisfies GD, this implies that M has height one, so R is one-dimensional. \square

To deal with the Mori case, we need the following lemma; we give a proof for lack of a reference.

Lemma 5.4. *Let I be a divisorial ideal of a Mori domain R . Then the domain $(I : I)$ is Mori.*

Proof. Let $J_1 \subseteq J_2 \subseteq \dots$ an infinite increasing sequence of divisorial ideals of the domain $(I : I)$. Since I is a divisorial ideal of R , the domain $(I : I)$ is a fractional divisorial ideal of R , so J_1, J_2, \dots are fractional divisorial ideals of R . Let c be a nonzero element of I . Then $cJ_1 \subseteq cJ_2 \subseteq \dots$ is an increasing sequence of divisorial ideals of R , so $cJ_n = cJ_{n+1}$ for $n \gg 0$. Thus the sequence $J_1 \subseteq J_2 \subseteq \dots$ stabilizes, implying that $(I : I)$ is Mori. \square

Proposition 5.5. *Let (R, M) be a finitely stable local Mori domain. If T is a finite extension of R , then R is one-dimensional, stable and every ideal of R is 2-generated, thus the domain R is Noetherian. (see 2.1 for the definition of T).*

Proof. By Corollary 2.21 (4), the maximal ideal M of R is divisorial and stable. We use the setting of Theorem 2.2. The domain R satisfies condition (a) of this theorem by Proposition 2.4, so $T = R_n$ for some integer $n \geq 0$. By Lemma 5.4, the domain $R_1 = (M : M)$ is Mori. By induction, R_k is Mori for all $k \geq 0$, so $T = R_n$ is a Mori domain. Thus T is Archimedean, so R is one-dimensional by Proposition 4.9. By Proposition 2.22, R is stable. By Theorem 2.5, every ideal of R is 2-generated. \square

Proposition 5.6. *Let (R, M) be a local domain. The following conditions are equivalent:*

- (i) *R is one-dimensional and stable.*
- (ii) *R is finitely stable and Mori.*

Proof. (i) \Rightarrow (ii) See Proposition 2.11.

(ii) \Rightarrow (i) By Corollary 2.21 (4), the maximal ideal M of R is divisorial and stable. By Proposition 5.5, we have to consider just the case (b) of Theorem 2.2. In this case, by Corollary 4.10, R is one-dimensional. By Proposition 2.22, R is stable. \square

In the next theorem we globalize Proposition 5.6:

Theorem 5.7. *Let R be an integral domain. The following two conditions are equivalent:*

- (i) *R is one-dimensional and stable.*
- (ii) *R is finitely stable and Mori.*

Moreover, if R satisfies these two equivalent conditions, then every overring of R also satisfies the two conditions, every overring of R is 2- v -generated, and R' is a Dedekind domain.

Proof. (i) \Rightarrow (ii) Since R is locally stable, we obtain that R is locally Mori by Proposition 5.6. Since R has finite character, it follows that R is Mori.

(ii) \Rightarrow (i) Since R is locally finitely stable and locally Mori, it follows that R is one-dimensional by Proposition 5.6.

Assume that R satisfies the two conditions. Let D be an overring of R . Since R is one-dimensional and R' is Prüfer (as R is stable), it follows that each overring of R is one-dimensional by [11, Theorem 6]. Since R is stable, each overring of R is stable. A one-dimensional stable domain is 2- v -generated by Proposition 2.18. Finally, R' is Prüfer and Mori, so it is Dedekind (alternatively, this follows from that a stable one-dimensional Prüfer domain is Dedekind). \square

In connection with Theorem 5.7, recall that an integral domain is Noetherian 2-generated if and only if it is one-dimensional, stable and divisorial ([22, Theorem 3.1] and [5, Theorem 7.3]).

However, if we assume just that R is a 2- v -generated domain, then R is not necessarily one-dimensional, and so also not finitely stable. Indeed, any Krull domain is 2- v -generated [19, Proposition 1.2]. In addition, it is

not true that in a 2- v -generated domain each divisorial ideal is stable. In fact, if R is a Krull domain, stability coincides with invertibility. Thus each divisorial ideal of a Krull domain R is stable (i.e., invertible) if and only if R is locally factorial [6, Lemma 1.1]. On the other hand, a one-dimensional Krull domain is Dedekind and so each nonzero ideal is divisorial and stable.

In view of this example and of the 2-generated case, we ask:

Question 5.8. *Let R be a 2- v -generated domain R . Are the divisorial ideals of R v -stable? If R is one-dimensional, are the divisorial ideals of R stable?*

Recall that an ideal I of a domain R is v -invertible if $(I(R : I))_v = R$ and that a divisorial ideal I of R is v -stable if I is v -invertible in the ring $(I : I)$, that is $(I(I : I^2))_v = (I : I)$.

We now turn to the question how to obtain an Archimedean stable local domain (R, M) of dimension greater than one. Here we use again B. Olberding's work, and also a useful suggestion of W. Heinzer.

If R is such a domain, with the usual notation 2.1, by Corollary 4.10, T is not local and so R must satisfy condition (a) of Theorem 2.2, that is, $T = R_n$ for some $n \geq 0$. Since R is stable, $T = R'$ is a Prüfer domain and T has exactly 2 maximal ideals, which we denote by N_1 and N_2 . Since R is Archimedean, T has a height-one maximal ideal by Proposition 4.7. We may assume that $\text{height } N_1 = 1$ and $\text{height } N_2 > 1$. Let $T = R_k$ with minimal $k \geq 0$, so $k > 0$ since T is not local. Thus R_{k-1} is local and, since any overring of a stable domain is stable [24, Theorem 5.1], R_{k-1} is stable. By Corollary 4.8, R_{k-1} is Archimedean. Also, $\dim R_{k-1} = \dim R > 1$. Replacing R by R_{k-1} , we may assume that $R_1 = T$.

We have canonical isomorphisms $R/M \cong T/N_i$ for $i = 1, 2$, so $T = R + N_1 = R + N_2$. In Example 6.17, k is a subfield of R canonically identified with R/M , so $T = k + N_1 = k + N_2$ and $M = N_1 \cap N_2$.

Lemma 5.9. *Let D be an integral domain. The following conditions are equivalent:*

- (i) D is Prüfer and it has exactly two maximal ideals.
- (ii) D is an intersection of two valuation domains.

Proof. (i) \Rightarrow (ii) Let Q_1 and Q_2 be the maximal ideals of D . Then $D = D_{Q_1} \cap D_{Q_2}$ is an intersection of two valuation domains.

(ii) \Rightarrow (i) See [18, Theorem 12.2]. □

Lemma 5.10. *Let $R \subseteq D$ be an extension of domains such that $D = R + xR$ for some element $x \in D$. Then D is a quadratic extension of R .*

Proof. Let $s = s_0 + s_1x, t = t_0 + t_1x$ be two elements in D , where $s_i, t_i \in R$ for $i = 1, 2$. Let I be the ideal $s_1R + t_1R$ of R . Thus $R + sR + tR = R + xI$. We have

$st \in (R + xI)^2 = (R + xI)R + (R + xI)xI \subseteq R + xI + (R + xR)I = R + xI$, implying that D is a quadratic extension of R . □

We use the following lemma of Olberding:

Lemma 5.11. *Let R be a finitely stable domain. If I is a nonzero ideal of R such that IR' is principal, then I is principal in $(I : I)$, in particular I is stable.*

Proof. By Theorem 1.1, R' is a quadratic extension of R and has at most two maximal ideals. Hence we can apply [27, Proposition 3.6]. \square

For the next proposition cf. [16, Theorem 14]. (The statement in the proof of [16, Theorem 14], that $u - u^2 \in R$ for each nonunit $u \in A$, is false in general, but this error can be easily corrected.)

Proposition 5.12. *Let (V_1, Q_1) and (V_2, Q_2) be two valuation domains with no inclusion relation among them, with principal maximal ideals, with the same field of fractions L , containing a field k , and such that $V_i = k + Q_i$, for $i = 1, 2$. Let $D = V_1 \cap V_2$, $N_i = Q_i \cap D$, for $i = 1, 2$, $M = N_1 \cap N_2$, $R = k + M$, and $R' = (M : M)$. Then:*

- (1) N_1, N_2 are the only maximal ideals of D , $N_1 \neq N_2$, and N_1, N_2 are principal. We have $D_{N_i} = V_i$ for $i = 1, 2$, so D is Prüfer, $\text{Frac } D = L$, and $D = k + N_i$ for $i = 1, 2$.
- (2) If $N_1 = xD$, then $D = R + xR$, so D is a 2-generated R -module. Moreover, M is a principal ideal of D and a 2-generated ideal of R . Also D is a quadratic extension of R and $D = R' = R_1$.
- (3) $R = k + M$ is a local domain with maximal ideal M .
- (4) The domain R is finitely stable with stable maximal ideal M .
- (5) R is stable if and only if D is stable, equivalently D is strongly discrete.
- (6) R is Archimedean if and only if one of the two valuation domains V_1, V_2 is one-dimensional, and so a DVR.
- (7) $\dim R > 1$ if and only if $\dim V_i > 1$ for some $i = 1, 2$.

Proof. (1) By [18, Theorem 12.2], N_1 and N_2 are the only maximal ideals of D , $N_1 \neq N_2$, and $D_{N_i} = V_i$ for $i = 1, 2$. For $i = 1, 2$, the maximal ideal N_i of D is locally principal, and so it is principal since D is semilocal.

For $i = 1, 2$ we have natural isomorphisms $D/N_i \cong D_{N_i}/N_i D_{N_i} = V_i/Q_i \cong k$, implying that $D = k + N_i$.

(2) Since the ideals N_1, N_2 of D are principal, we deduce that also $M = N_1 N_2$ is a principal ideal of D . Thus D is an overring of R , and $D = (M : M) = R_1$.

Since $xN_2 \subseteq N_1 N_2 = M$, we have:

$$D = k + N_1 = k + xD = k + x(k + N_2) = k + xN_2 + xk \subseteq R + kx.$$

Hence $D = R + xR$, implying by Lemma 5.10 that D is a quadratic extension of R . Since D is a Prüfer domain, and D is a quadratic, so integral, overring of R , it follows that $D = R'$.

As M is a principal ideal of D and $D = R + Rx$ is a 2-generated R -module, it follows that M is a 2-generated ideal of R .

(3) By definition, $R = k + M$, so M is a maximal ideal of R . If P is a maximal ideal of R then $P = N_i \cap R$ for some integer $i = 1, 2$, because $D = R'$ by (2). Thus $M = (N_1 \cap R) \cap (N_2 \cap R) \subseteq P$, implying that $M = P$. Thus (R, M) is a local domain.

(4) By item (1), $D = R'$ is Prüfer with two maximal ideals and by item (2), D is a quadratic extension of R . By Olberding's characterization 1.1, R is finitely stable. Since M is a principal, so stable, ideal of D and $\text{Frac } R = \text{Frac } D$, it follows that M is a stable ideal of R .

(5) If R is stable then D is stable, since each overring of a stable domain is stable.

Conversely, assume that D is stable. By item (2), we have $D = R' = R_1$ and $M = mR'$, where $m \in M$.

Let I be a nonzero ideal of R , and let $A = (I : I)$.

The domain D is Prüfer, and, as shown at the end of the proof of [24, Theorem 4.2], $D = R_1$ is a minimal overring of R . Hence by [14, Proposition 2.4 and Terminology on page 137], D is contained in every overring of R that is different from R . Hence, either $A = R$, or $D \subseteq A$. If $D \subseteq A$, then $A = (I : I)$ is a stable domain, so the ideal I of A is invertible in $(I : I)$, implying that I is a stable ideal of R .

Now assume that $A = R$. Since M is a principal ideal of R' , it follows that $(IR' : IR') = (IM : IM)$. Also $(IM : IM)MI \subseteq IM \subseteq I$, so $(IM : IM)M \subseteq (I : I) = R$. Hence $(IR' : IR') \subseteq (R : M)$. If $(R : M) \neq (M : M)$, then the maximal ideal M of the local domain R is invertible, so principal, implying that $R = R_1 = (M : M)$, a contradiction. If $(R : M) = (M : M)$, then $(IR' : IR') = R'$. Hence IR' is an invertible, so principal, ideal of R' since R' is stable and semilocal. By Lemma 5.11, I is a stable ideal.

(6) Since R is local finitely stable, and $D = R'$, this follows from Proposition 4.7.

(7) Indeed, $\dim R = \dim D = \max(\dim V_1, \dim V_2)$. □

Corollary 5.13. *Let (V_1, Q_1) and (V_2, Q_2) be two strongly discrete valuation domains with no inclusion relation among them, with principal maximal ideals, with the same field of fractions L , containing a field k , and such that $V_i = k + Q_i$, for $i = 1, 2$. Let $\dim V_1 = 1$ and $\dim V_2 = n$, where $2 \leq n \leq \infty$. Let $D = V_1 \cap V_2$, $N_i = Q_i \cap D$, for $i = 1, 2$, $M = N_1 \cap N_2$, and $R = k + M$.*

Then R is an n -dimensional Archimedean stable local domain.

Moreover, we have:

- (1) *R satisfies accp, but R' is not Archimedean.*
- (2) *The pair (R, D) does not satisfy GD (the going down property).*

Proof. By Proposition 5.12, R is an n -dimensional Archimedean stable local domain since D is a strongly discrete Prüfer domain.

(1) By Corollary 3.20, any Archimedean stable domain satisfies accp. By Corollary 3.2, R' is not Archimedean since R' is semilocal of dimension greater than 1.

(2) R does not satisfy GD by Proposition 5.3. \square

6. EXAMPLES

It is well-known that the accp and the Archimedean properties do not localize. In [15, Example 2] Anne Grams constructs a one-dimensional Prüfer domain of finite character which satisfies accp (the ascending chain condition on principal ideals) and each of its localizations but one is a DVR, while the other one is a valuation domain that is not a DVR, so it does not satisfy accp (see comments and more examples in [1] and its references). Also, [15] (page 328) provides a general construction of an almost Dedekind domain A with accp whose Nagata ring $A(X)$ is not an accp domain (so that $A[X]$ is accp, while its localization $A(X)$ is not accp). This example as well as [15, Example 2] is one-dimensional, so it is locally Archimedean.

The ring of entire functions E is an infinite-dimensional completely integrally closed (hence Archimedean) Bézout domain [7, Section 8.1], but it is not locally Archimedean since the localizations at maximal ideals are valuation domains, and a valuation domain that is not a field is Archimedean if and only if it is one-dimensional. The ring E does not satisfy accp and it does not have finite character: for example, if f is a nonzero entire function with infinitely many zeros c_1, c_2, \dots (e.g., $\sin z$), then $f \in \bigcap_{n=1}^{\infty} \prod_{i=1}^n (Z - c_i)$, so the domain E does not satisfy accp, and E does not have finite character since f belongs to the maximal ideals $(Z - c_i)E$ for all i .

We construct in Example 6.9 below a completely integrally closed (for short, c.i.c.) domain R satisfying accp with only two maximal ideals such that R_M is not Archimedean for each maximal ideal M of R , thus R_M does not satisfy accp. Of course, R is Archimedean and has finite character. We construct first a c.i.c. local domain A with accp such that A_P is not Archimedean for some prime ideal P (Example 6.8). Then we “double” this construction to obtain Example 6.9 (see Remark 6.10).

We also construct a stable Prüfer domain R with accp that is not locally Archimedean (Example 6.13), thus the converse of Corollary 3.20 is false.

In Example 6.14 we construct a local one-dimensional domain R such that R' is a finite extension of R , the ring R' is a PID, so stable, but R is not even finitely stable (cf. Proposition 5.12 (5) and Lemma 5.11).

In Example 6.15 we construct a stable valuation domain with prime spectrum consisting of an infinite descending chain of prime ideals following Olberding ([21, Proposition 5.4]). We use this example in the last Example 6.17, where we present a stable Archimedean local domain of arbitrary dimension.

Recall that a set of subrings \mathbf{S} of a ring R is *directed* if for each $A, B \in \mathbf{S}$ there exists $C \in \mathbf{S}$ such that both A and B are contained in C .

Lemma 6.1. *Let R be an integral domain that is a directed union of a set \mathbf{S} of c.i.c. subrings. Assume $A = R \cap \text{Frac}(A)$ for each $A \in \mathbf{S}$. Then R is c.i.c..*

Proof. Assume for $f \in R \setminus \{0\}$ and $g \in \text{Frac}(R)$ that $fg^n \in R$ for all $n \geq 1$. Since the union of the subrings in \mathbf{S} is directed, there exists a domain $A \in \mathbf{S}$ such that $f \in A$ and $g \in \text{Frac}(A)$. Hence $fg^n \in R \cap \text{Frac}(A) = A$, for all $n \geq 1$. Since A is c.i.c., we obtain that $g \in A \subseteq R$. Thus R is c.i.c. \square

Lemma 6.2. *Let R be an integral domain that is a directed union of a set \mathbf{S} of accp subrings. Assume that for each $A \in \mathbf{S}$ there exists a retraction $\varphi_A : R \rightarrow A$ mapping nonunits of R to nonunits of A . Then R satisfies accp.*

Proof. Assume that R does not satisfy accp. Hence there exists a strictly increasing infinite sequence of nonzero principal ideals in R :

$$r_1 R \subsetneq r_2 R \subsetneq r_3 R \subsetneq \dots$$

We have $r_1 \in A$ for some domain $A \in \mathbf{S}$. Let $\varphi = \varphi_A$. Since $r_1 \neq 0$, there is an increasing sequence of nonzero principal ideals in the ring A :

$$r_1 A = \varphi(r_1) A \subseteq \varphi(r_2) A \subseteq \varphi(r_3) A \subseteq \dots$$

For each $n \geq 1$, we have $\frac{r_n}{r_{n+1}} \in R \setminus \mathcal{U}(R)$; hence $\varphi\left(\frac{r_n}{r_{n+1}}\right) = \frac{\varphi(r_n)}{\varphi(r_{n+1})} \in A \setminus \mathcal{U}(A)$. It follows that all the inclusions in the sequence

$$\varphi(r_1) A \subseteq \varphi(r_2) A \subseteq \varphi(r_3) A \subseteq \dots$$

are strict, contradicting the assumption that A satisfies accp. \square

Proposition 6.3. *Let if $\varphi : A \rightarrow B$ be an homomorphism of rings. Consider the following two conditions:*

- (1) φ maps nonunits to nonunits.
- (2) $\ker \varphi \subseteq \text{Jac}(A)$.

Then (1) \Rightarrow (2). If φ is surjective, then the two conditions are equivalent. In particular, if A is local, then any surjective homomorphism $\varphi : A \rightarrow B$ maps nonunits to nonunits.

Proof. (1) \Rightarrow (2) Let $c \in \ker \varphi$. Assume that $c \notin \text{Jac}(A)$. Since φ is surjective, there exists an element $a \in A$ such that $1 + ac$ is not a unit in A , although $\varphi(1 + ac) = 1$, a contradiction.

(2) \Rightarrow (1) assuming that φ is surjective. Assume that for some nonunit $c \in A$, the element $\varphi(c)$ is invertible in B . Since φ is surjective, there exists an element $a \in A$ such that $\varphi(c)\varphi(a) = 1$. Hence $\varphi(1 - ca) = 0$, so $1 - ca \in J(A)$, implying that ca is invertible in A . Thus c is invertible in A , a contradiction. \square

Proposition 6.4. *Let R be an integral domain that is a directed union of a set \mathbf{S} of c.i.c. subrings satisfying accp. Assume that for every $A \in \mathbf{S}$ there exists a retraction $\varphi_A : R \rightarrow A$ mapping nonunits of R to nonunits of A . Then R is c.i.c. and it satisfies accp.*

Proof. The domain R satisfies accp by Lemma 6.2.

For $A \in \mathbf{S}$ we have $A = R \cap \text{Frac}(A)$, since A is a retract of R . Thus R is c.i.c. by Lemma 6.1. \square

Corollary 6.5. *Let R be an integral domain that is a directed union of a set \mathbf{S} of integrally closed Noetherian subrings. Assume that for every $A \in \mathbf{S}$ there exists a retraction $\varphi_A : R \rightarrow A$ mapping nonunits of R to nonunits of A . Then R is c.i.c. and it satisfies accp.*

Proof. Indeed, a Noetherian ring satisfies accp, and an integrally closed Noetherian domain is c.i.c. Hence the corollary follows from Proposition 6.4. \square

Lemma 6.6. *Let A be an integrally closed domain, let $n \geq 1$ and let X, Y, Z_i ($1 \leq i \leq n$) be independent indeterminates over A . Then the domain*

$$D = A[X, Y, Z_i, \frac{XZ_i}{Y^i} \ (1 \leq i \leq n)]$$

is integrally closed.

Proof. Let S be the multiplicative monoid generated by $X, Y, Z_i, \frac{XZ_i}{Y^i}$ ($1 \leq i \leq n$). We show that the monoid S is integrally closed. Let G be the group of fractions of S , that is, G is the multiplicative group generated by X, Y, Z_i ($1 \leq i \leq n$). Let g be an element of G such that $g^k \in S$ for some integer $k \geq 1$. Since the monoid generated by $X, Y, \frac{1}{Y}, Z_i$ ($1 \leq i \leq n$) is integrally closed, it follows that g belongs to this monoid. Thus

$$g = X^f Y^m \prod_{i=1}^n Z_i^{r_i},$$

where f, r_i are nonnegative integers for all i , and m is an integer. We have

$$(2) \quad g^k = X^{kf} Y^{km} \prod_{i=1}^n Z_i^{kr_i} = X^a Y^b \prod_{i=1}^n Z_i^{c_i} \prod_{i=1}^n \left(\frac{XZ_i}{Y^i} \right)^{e_i},$$

where a, b, c_i, e_i are nonnegative integers for all i . We may assume that the sum $a + \sum_{i=1}^n ic_i$ is minimal.

First assume that $c_i = 0$ for all i . Comparing exponents of the indeterminates Z_i on the two sides of (2), we obtain that $e_i = kr_i$ for all i , so a and b are divisible by k . It follows that $g \in S$.

Now assume that $c_{i_0} > 0$ for some index i_0 . If $a > 0$, then

$$g^k = X^{a-1} Y^{b+i_0} \left(Z_{i_0}^{c_{i_0}-1} \prod_{i \neq i_0} Z_i^{c_i} \right) \left(\frac{XZ_{i_0}}{Y^{i_0}} \prod_{i=1}^n \frac{X^{e_i} Z_i^{e_i}}{Y^{ie_i}} \right),$$

contradicting the minimality of $a + \sum_{i=1}^n ic_i$. Thus $a = 0$.

Let j, q be integers such that $c_j > 0$ and $e_q > 0$. If $j > q$, we interchange Z_j and Z_q as follows:

$$g^k = Y^{b+j-q} \left(Z_q Z_j^{c_j-1} \prod_{i:i \neq j} Z_i^{c_i} \right) \left(\frac{X Z_j}{Y^j} \left(\frac{X Z_q}{Y^q} \right)^{e_q-1} \prod_{i:i \neq q} \left(\frac{X Z_i}{Y^i} \right)^{e_i} \right),$$

contradicting the minimality assumption on $a + \sum_{i=1}^n i c_i$. Hence $j \leq q$ for all j and q such that c_j and e_q do not vanish. We have

$$(3) \quad g^k = X^{kf} Y^{km} \prod_{i=1}^n Z_i^{kr_i} = Y^b \prod_{i=1}^n Z_i^{c_i} \prod_{i=1}^n \left(\frac{X Z_i}{Y^i} \right)^{e_i}.$$

Let $1 \leq q \leq n$ be an integer such that $q \neq q_0 = \min_{e_i > 0} i$. Since either $c_q = 0$ or $e_q = 0$, and since by (3) we have $c_q + e_q = kr_q$, it follows that both c_q and e_q are divisible by k . Comparing the exponents of X on both sides of (3), since all c_i, e_i for $i \neq q_0$ are divisible by k , we see that also e_{q_0} is divisible by k . Clearly, also c_{q_0} and b are divisible by k . Thus $g \in S$, so the monoid S is integrally closed. By [13, Corollary 12.11 (2)], the domain D is integrally closed. \square

Remark 6.7. *The domain D in Lemma 6.6 is isomorphic to a subring of a polynomial ring over the domain A in $n+2$ indeterminates. Indeed, for $U_i = \frac{Z_i}{Y^i}$ ($0 \leq i \leq n$) we have $D = A[X, Y, XU_i, Y^i U_i \ (1 \leq i \leq n)] \subseteq k[X, Y, U_i \ (0 \leq i \leq n)]$. Similarly, the domains D of Example 6.8 and A of Example 6.9 below may be viewed as subrings of a polynomial ring over k in infinitely many indeterminates.*

Example 6.8. *A completely integrally closed local domain R with accp such that R_P is not Archimedean for some prime ideal P .*

Let k be a field and let

$$D = k[X, Y, Z_n, \frac{X Z_n}{Y^n} \ (n \geq 1)],$$

where X, Y, Z_n ($n \geq 1$) are independent indeterminates over k . Let M be the maximal ideal of D generated by the elements $X, Y, Z_n, \frac{X Z_n}{Y^n}$ ($n \geq 1$). Set

$$R = D_M \text{ and } P = \langle X, Y, \frac{X Z_n}{Y^n} \ (n \geq 1) \rangle R.$$

For each $n \geq 1$, let $D_n = k[X, Y, Z_i, \frac{X Z_i}{Y^i} \ (1 \leq i \leq n)]$ and $R_n = (D_n)_{M_n}$, where M_n is the maximal ideal of D_n generated by $X, Y, Z_i, \frac{X Z_i}{Y^i}$ ($1 \leq i \leq n$), thus $M_n = M \cap D_n$.

Clearly $R_1 \subseteq R_2 \subseteq \dots$ and $R = \bigcup_n R_n$. For each n , there exists a retraction $\varphi_n : R \rightarrow R_n$ that maps to 0 each indeterminate Z_i , for $i > n$. Clearly $\varphi_n(MR) \subseteq M_n R_n$. By Lemma 6.6, the domains R_n are integrally closed. Since the domains R_n are Noetherian, from Corollary 6.5 it follows that R is c.i.c. and R satisfies accp.

The ideal P is prime since P is the set of all rational functions in R vanishing when plugging in first $X = 0$, and then $Y = 0$ (thus these rational functions are defined for $X = 0$, and *after* plugging in $X = 0$, we obtain a function defined for $Y = 0$). For all $n \geq 1$, the elements Z_n are invertible in R_P , so $\frac{X}{Y^n} \in R_P$. Since Y is not invertible in R_P , we see that the domain R_P is not Archimedean. \square

Example 6.9. *A completely integrally closed domain R satisfying accp with just two maximal ideals such that for each maximal ideal M , the domain R_M is not Archimedean.*

Let k be a field and let

$$A = k[X_1, Y_1, Z_{1,n}, \frac{X_1 Z_{1,n}}{Y_1^n}; X_2, Y_2, Z_{2,n}, \frac{X_2 Z_{2,n}}{Y_2^n} \ (n \geq 1)],$$

where $X_i, Y_i, Z_{i,n} (i = 1, 2, n \geq 1)$ are independent indeterminates over k .

Let

$$P_1 = \langle X_1, Y_1, \frac{X_1 Z_{1,n}}{Y_1^n}, Z_{2,n}, \frac{X_2 Z_{2,n}}{Y_2^n} \ (n \geq 1) \rangle A \text{ and}$$

$$P_2 = \langle X_2, Y_2, \frac{X_2 Z_{2,n}}{Y_2^n}, Z_{1,n}, \frac{X_1 Z_{1,n}}{Y_1^n} \ (n \geq 1) \rangle A.$$

The ideal P_1 is prime since it is the set of all rational functions in A vanishing when plugging in first $X_1 = Z_{2,n} = 0$ for all n , and then $Y_1 = 0$. Similarly, the ideal P_2 is prime.

For all $n \geq 1$, the elements $Z_{1,n}$ are invertible in A_{P_1} , so $\frac{X_1}{Y_1^n} \in A_{P_1}$. Since Y_1 is not invertible in A_{P_1} , we see that the domain A_{P_1} is not Archimedean. Similarly, the domain A_{P_2} is not Archimedean.

Let $S = A \setminus (P_1 \cup P_2)$, and $R = A_S$, thus $R = A_{P_1} \cap A_{P_2}$. Hence R has just two maximal ideals, namely $M_1 = P_1 A_{P_1} \cap R$ and $M_2 = P_2 A_{P_2} \cap R$. We have $R_{M_i} = A_{P_i}$ for $i = 1, 2$, so the domains R_{M_1} and R_{M_2} are not Archimedean.

For each $n \geq 1$, let

$$A_n = k[X_1, Y_1, Z_{1,j}, \frac{X_1 Z_{1,j}}{Y_1^j}; X_2, Y_2, Z_{2,j}, \frac{X_2 Z_{2,j}}{Y_2^j} \ (1 \leq j \leq n)]$$

and $R_n = (A_n)_{S_n}$, where $S_n = S \cap A_n$.

By Lemma 6.6, the domains

$$D_n = k[X_1, Y_1, Z_{1,j}, \frac{Z_{1,j} X_1}{Y_1^j} \ (1 \leq j \leq n)]$$

and $A_n = D_n[X_2, Y_2, Z_{2,j}, \frac{X_2 Z_{2,j}}{Y_2^j} \ (1 \leq j \leq n)]$ are integrally closed. Hence R_n is integrally closed.

Clearly $R_1 \subseteq R_2 \subseteq \dots$ and $R = \bigcup_n R_n$. For each $n \geq 1$ we have a retraction $\varphi_n : R = A_S \rightarrow R_n$ that maps to 0 each indeterminate $Z_{i,j}$ for $i = 1, 2$ and $j > n$ since the elements $Z_{1,j}$ and $Z_{2,j}$ do not belong to S .

Clearly the elements in $\varphi_n(M_1 \cup M_2)$ are nonunits in R_n . Since the domains R_n are Noetherian and integrally closed, it follows from Corollary 6.5 that R is c.i.c. and R satisfies accp. \square

Of course, the domain R in Example 6.9 is not Mori, since any localization of a Mori domain is Mori and so Archimedean.

Remark 6.10. *If D and A are the domains defined in Examples 6.8 and 6.9, respectively, then $A \cong D \otimes_k D$.*

The next example 6.13 shows that a stable Archimedean domain may not be locally Archimedean. We will use below the following well-known facts:

Lemma 6.11. *(see [7, Lemma 1.1.4 and Proposition 5.3.3]) Let U be a valuation domain (possibly a field), let $K = \text{Frac}(U)$, and let X be an indeterminate over U . Then $V = U + XK[X]_{\langle X \rangle}$ is a valuation domain. If U is strongly discrete, then also V is strongly discrete. The prime ideals of V are all the ideals $P + XK[X]_{\langle X \rangle}$, where P is a prime ideal of U . Moreover, if P is nonzero, then $P + XK[X]_{\langle X \rangle} = PV$ and $(P + XK[X]_{\langle X \rangle}) \cap U = P$. For $P = (0)$ the ideal $XK[X]_{\langle X \rangle}$ is the least nonzero prime ideal of V . Thus, if U is finite dimensional, then $\dim V = \dim U + 1$.*

Corollary 6.12. *Let X and Y be two independent indeterminates over a field k , let $C = k[Y, \frac{X}{Y^n} \ (n \geq 1)]$, and let P be the maximal ideal $YC = \langle X, \frac{X}{Y^n} \ (n \geq 1) \rangle$ of C . Then $V = C_P$ is a strongly discrete 2-dimensional valuation domain.*

Proof. Clearly, $V = k[Y]_{\langle Y \rangle} + Xk(Y)[X]_{\langle X \rangle}$. By Lemma 6.11, V is a strongly valuation domain of dimension 2. \square

Example 6.13. *A stable 2-dimensional Prüfer domain R satisfying accp with just two maximal ideals of height 2. Thus for each maximal ideal M of R , except the two maximal ideals of height 2, the domain R_M is a DVR. Also R is Archimedean, but not locally Archimedean: R_M is not Archimedean if M is a maximal ideal of R of height 2.*

Let X and Y be two independent indeterminates over a field k . Set

$$R = k[X, Y, \frac{X(1-X)^n}{Y^n}, \frac{Y^{n+1}}{(1-X)^n} \ (n \geq 1)]_S,$$

where $S = k[Y] \setminus Yk[Y]$.

Let $T = \frac{1-X}{Y}$. We have $X = 1 - YT$, so

$$R = k[Y, YT, (1 - YT)T^n, \frac{Y}{T^n} \ (n \geq 1)]_S$$

(as shown in item (1) below, R satisfies accp, thus R is Archimedean, although $\frac{Y}{T^n} \in R$ for all $n \geq 1$. This is not a contradiction since $T \notin R$).

- (1) *R satisfies accp.*

Let f and g_n ($n \geq 1$) be nonzero elements of R such that $\frac{f}{\prod_{i=1}^n g_i} \in R$ for all $n \geq 1$. To prove that g_i is a unit for $i \gg 0$, we may assume that $g_i \in k[Y, YT, (1 - YT)T^n, \frac{Y}{T^n}]$ ($n \geq 1$), for all $i \geq 1$.

Since the elements Y and T are algebraically independent over k , we may view the ring $k[Y, YT, (1 - YT)T^n, \frac{Y}{T^n}]$ ($n \geq 1$) as a subring of the polynomial ring $k(T)[Y]$. Thus for $i \gg 0$ we have $\deg_Y(g_i) = 0$, that is, $g_i \in k(T)$.

For $i \gg 0$, since $g_i \in k[Y, YT, (1 - YT)T^n, \frac{Y}{T^n}]$ ($n \geq 1$), by plugging in $Y = 0$, we obtain that $g_i \in k[T]$; by plugging in $Y = \frac{1}{T}$, we obtain that $g_i \in k[\frac{1}{T}]$, so $g_i \in k[T] \cap k[\frac{1}{T}] = k$. We conclude that R satisfies accp.

- (2) *R is a stable 2-dimensional Prüfer domain with just two maximal ideals of height 2.*

Let M be a maximal ideal of R .

- (3) *Assume that $Y \notin M$. Then:*

- R_M is a DVR, so $\text{height } M = 1$.
- Each nonzero element of R belongs to just finitely many maximal ideals of R not containing Y .

Clearly $R \subseteq D = k(Y)[X, \frac{1}{1-X}] \subseteq R_M$, and R_M is a ring of fractions of D . Hence R_M is a local PID, that is, a DVR.

For each maximal ideal M of R not containing Y we have $MR_M = PD_P$ for $P = M \cap D$, and since D is a PID, each nonzero element of R belongs to just finitely many prime ideals of D , and so it belongs to just finitely many maximal ideals of R not containing Y .

- (4) *Assume that $Y \in M$. Then R_M is a stable 2-dimensional valuation domain, in particular $\text{height } M = 2$.*

Since $X(1 - X) \in RY \subseteq M$ it follows that either $X \in M$ or $1 - X \in M$.

- (a) *Assume that $Y, X \in M$.*

Clearly, $C = k[Y, \frac{X}{Y^n}]$ ($n \geq 1$) $\subseteq R_M$. Since the maximal ideal $P = \langle Y, \frac{X}{Y^n} \mid n \geq 1 \rangle$ of C is contained in MR_M , it follows that $P = MR_M \cap C$. Since $R \subseteq C[\frac{1}{1-X}] \subseteq C_P \subseteq R_M$, it follows that $C_P = R_M$. By Corollary 6.12, $R_M = C_P$ is a 2-dimensional strongly discrete, and so stable, valuation domain. Also M is uniquely determined by the requirement $Y, X \in M$, namely $M = PC_P \cap R$.

- (b) *Assume that $Y, 1 - X \in M$.*

Recall that $T = \frac{1-X}{Y}$. Since $XT = \frac{X(1-X)}{Y} \in R$ and X is a unit in R_M , we see that $T \in R_M$. Hence

$$\tilde{C} = k[T, \frac{Y}{T^n} \mid n \geq 1] \subseteq R_M,$$

the maximal ideal $\tilde{P} = \langle T, \frac{Y}{T^n} \ (n \geq 1) \rangle$ of the ring \tilde{C} is contained in MR_M , and $R \subseteq \tilde{C}_{\tilde{P}} \subseteq R_M$. As in item (b) (i), we conclude that $R_M = \tilde{C}_{\tilde{P}}$ is a 2-dimensional strongly discrete, and so stable, valuation domain and that M is uniquely determined by the requirement $Y, 1 - X \in M$.

Thus R has finite character and each localization of R at a maximal ideal is a stable valuation domain. Hence R is a stable Prüfer domain [24, Theorem 3.3].

We have also proved that R is 2-dimensional with exactly 2 maximal ideals of height 2. The localizations at these two maximal ideals are not Archimedean, as seen directly from the above proof. Actually, as it is well-known, a valuation domain is Archimedean if and only if it is one-dimensional. The maximal ideals of R are invertible since R is stable and Prüfer. Thus in Corollary 3.10 (2) we may not assume just that R is Archimedean rather than locally Archimedean.

Example 6.13 shows that the converse of Corollary 3.20 is false: a stable domain R which satisfies accp need not be locally Archimedean, even if R is Prüfer and 2-dimensional.

Example 6.14. *A local integral domain (R, M) with the following properties:*

- (1) *R is one-dimensional, Noetherian, not (finitely) stable, but with stable maximal ideal.*
- (2) *$R' = (M : M)$ is a finitely generated R -module.*
- (3) *R' is a principal ideal local domain, so R' is stable and Prüfer.*

Let $K = \mathbb{Q}(\sqrt[3]{2})$. Let $R = \mathbb{Q} + XK[[X]]$. Thus $R' = K[[X]]$ is a principal ideal local domain with maximal ideal $M = XK[[X]]$, so R is one-dimensional, and R' is a 3-generated R -module. By the Eakin-Nagata Theorem, R is Noetherian. Clearly, R' is not a quadratic extension of R , so R is not finitely stable. Explicitly, the fractional ideal $I = \langle 1, \sqrt[3]{2} \rangle$ of R is not stable (equivalently, the ideal $\langle X, X\sqrt[3]{2} \rangle$ of R is not stable). Indeed, $I^2 = \langle 1, \sqrt[3]{2}, \sqrt[3]{4} \rangle$ and $(I : I^2) = XR$, so $I(I : I^2) = XR \neq R$. It follows that I is not stable. The maximal ideal M of R is stable, since M is an ideal of the stable domain R' which is an overring of R . \square

In the next example we present a well-known construction which is related to the construction in the proof of the Kaplansky-Jaffard-Ohm Theorem [8, Ch.III, Theorem 5.3]. This example illustrates explicitly a particular case of Olberding's Theorem [21, Proposition 5.4], and will be also used for Example 6.17.

Example 6.15. *For each $1 \leq n \leq \infty$ and for a field k , a strongly discrete, so stable, n -dimensional valuation domain V containing k . In particular, if $n = \infty$, the nonzero prime ideals of V form a descending infinite sequence, so the height of every nonzero prime ideal of V is infinite. Moreover, for all*

n , $\text{Frac } V$ is a purely transcendental extension of k of transcendence degree \aleph_0 .

First let $n = \infty$. Let $V = A_Q$, where

$$A = k[X_n, \frac{X_{n+1}}{X_n^i} \ (n \geq 1, i \geq 1)],$$

k is a field, X_n ($n \geq 1$) are independent indeterminates over k , and $Q = X_1 A = \langle X_n, \frac{X_{n+1}}{X_n^i} \ (n \geq 1) \rangle A$ is a maximal ideal of A .

It is easy to show that $V = \bigcup_{n=1}^{\infty} V_n$ (an ascending union), where V_n are subrings of V defined inductively as follows: $V_0 = k$, and for $n \geq 1$, we let $V_n = V_{n-1} + X_n (k(X_1, \dots, X_{n-1}[X_n])_{\langle X_n \rangle})$.

By induction, $\text{Frac}(V_n) = k(X_1, \dots, X_n)$ for $n \geq 1$. Thus $V_n = V_{n-1} + X_n (\text{Frac}(V_{n-1})[X_n]_{\langle X_n \rangle})$ for $n \geq 1$. Hence by Corollary 6.12, we obtain inductively that V_n is a strongly discrete valuation domain of dimension n , with maximal ideal $M_n = X_1 V_n$, and that the nonzero prime ideals of V_n form a descending chain

$$M_n = P_{n,n} \supsetneq P_{n,n-1} \supsetneq \dots \supsetneq P_{n,1}.$$

It follows that the domain $V = \bigcup_{n=1}^{\infty} V_n$ is a strongly discrete, so stable, valuation domain with maximal ideal $M = X_1 V$. Let P be a nonzero prime ideal of V . Since $P = \bigcup_{n=1}^{\infty} (P \cap V_n)$, we have $P \cap V_n \neq (0)$ for some integer $n \geq 1$. By Lemma 6.11, $P = (P \cap V_n)V = P_{n,i}V$ for an integer $1 \leq i \leq n$. If n is minimal, then $P_{n,i}$ is the least nonzero prime ideal of V_n , so $i = 1$. Hence the nonzero prime ideals of V form an infinite descending chain $M = P_1 \supsetneq P_2 \supsetneq \dots$, where $P_n = P_{n,1}V$ for all $n \geq 1$.

Thus for all $n \geq 1$, P_n is the ideal of V generated by the one-dimensional subspace $X_n k(X_1, X_2, \dots, X_{n-1})$ of V over the field $k(X_1, X_2, \dots, X_{n-1})$.

Explicitly, for all $n \geq 1$ we have

$$P_n = \sum_{i=n}^{\infty} X_i (k(X_1, \dots, X_{i-1})[X_i]_{\langle X_i \rangle}).$$

If n is finite, similarly to the definition of V above, we define $V_n = A_{Q_n}$, where

$$A = k[X_j, \frac{X_{j+1}}{X_j^i} \ (1 \leq j < n, i \geq 1)],$$

and $Q = X_1 A$ is a maximal ideal of A . (if $n = 1$, then $A = k[X_1]$). \square

In the last example we exhibit an n -dimensional Archimedean stable local domain, for each $n \geq 2$; thus answering in the negative the question posed in [9, Problem 7.1]. (For details concerning this example, see Propositions 5.12 and 5.13 above.)

We need the following lemma:

Lemma 6.16. *Let k be a field, and let $L \neq k$ be a purely transcendental field extension of k with $\text{tr.d. } L/k \leq \aleph_0$. Then there exists a DVR (V, N) such that $\text{Frac } V = L$ and $V/N = k$.*

Proof. Let $L = k(B)$, where B is a set of algebraically independent elements over k . Since $\text{tr.d. } L/k \leq \aleph_0 \leq \text{tr.d. } k((X))/k$ [17, Lemma 1, Section 3], there exists a subset B_0 of $k((X))$ containing X such that $|B_0| = |B|$. Thus there exists an isomorphism over k of the fields L and $k(B_0)$ mapping B onto B_0 . Hence we may assume that $L = k(B) \subseteq k((X))$ and that $X \in B$. Define $V = k[[X]] \cap L$. Thus V is a DVR with maximal ideal XV , and $V/XV \cong k$. Since $k[B] \subseteq V \subseteq L = k(B)$, it follows that $\text{Frac}(V) = L$. \square

Example 6.17. For $1 \leq n \leq \infty$, a stable n -dimensional Archimedean local domain (R, M) .

By Example 6.15, for any field k , there exists a stable n -dimensional valuation domain (V_2, Q_2) containing k such that $\text{Frac } V_2 = L$ is a purely transcendental extension of k and $V_2/Q_2 = k$. By Lemma 6.16, there exists a DVR (V_1, Q_1) containing k such that $\text{Frac}(V_1) = L$, and $V_1/Q_1 = k$. By Proposition 5.12, there exists a local Archimedean finitely stable domain R such that $R' = V_1 \cap V_2$ and by Proposition 5.13 such a domain is stable. \square

By Example 6.17 and by Proposition 5.13 (1), the integral closure of an Archimedean domain, or even an accp stable domain, is not necessarily Archimedean. The domain $\mathbb{Z} + X\overline{\mathbb{Z}}[X]$, where $\overline{\mathbb{Z}}$ is the ring of all algebraic integers, satisfies accp while $R' = \overline{\mathbb{Z}}[X]$ does not, although R' is Archimedean [2, Example 5.1].

REFERENCES

- [1] D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990), 1–19.
- [2] D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Rings between $D[X]$ and $K[X]$* , Houston J. Math. **17** (1991), 109–129.
- [3] D. D. Anderson, J. A. Huckaba and I. J. Papick, *A note on stable domains*, Houston J. Math. **13** (1987), 13–17.
- [4] V. Barucci, *Mori domains*, Non-Noetherian Commutative Ring Theory; Recent Advances, Chapter 3, Kluwer Academic Publishers, 2000.
- [5] S. Bazzoni and L. Salce, *Warfield domains*, J. Algebra **185** (1996), 836–868.
- [6] A. Bouvier, *The local class group of a Krull domain*, Canad. Math. Bull. **26** (1983), 13–19.
- [7] M. Fontana, J.A. Huckaba and I.J. Papick, *Prüfer domains*, Monographs and Textbooks in Pure and Applied Mathematics **203**, M. Dekker, New York, 1997.
- [8] L. Fuchs and L. Salce, *Modules over Non-Noetherian Domains*, Mathematical Surveys and Monographs **84**, American Mathematical Society, 2001.
- [9] S. Gabelli, *Ten problems on stability of domains*, Commutative Algebra - Recent Advances in Commutative Rings, Integer-valued Polynomials, and Polynomial functions, 175–193, Springer, New York, 2014.
- [10] S. Gabelli and G. Picozza, *Star stability and star regularity for Mori domains*, Rend. Semin. Mat. Univ. Padova, **126** (2011), 107–125.
- [11] R. Gilmer, *Domains in Which Valuation Ideals are Prime Powers*, Arch. Math., **17** (1966), 210–215.
- [12] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.
- [13] R. Gilmer, *Commutative semigroup rings*, University of Chicago, 1984.
- [14] R. Gilmer and W. Heinzer, *Intersections of quotients rings of an integral domain*, J. Math. Kyoto Univ. **7** (1967), 133–149.
- [15] A. Grams, *Atomic rings and the ascending chain condition for principal ideals*, Proc. Camb. Phil. Soc., **75** (1974), 321–329.
- [16] A. Jaballah, *Maximal non-Prüfer and maximal non-integrally closed subrings of a field*, J. Algebra Appl., **11** (2012), 877–895.
- [17] S. MacLane and O. F. G. Schilling, *Zero-Dimensional Branches of Rank One on Algebraic Varieties*, Ann. of Math., **401** (1939), 507–520.
- [18] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989.
- [19] J. L. Mott and M. Zafrullah, *On Krull domains*, Arch. Math. **56** (1991), 559–568.
- [20] J. Ohm, *Some Counterexamples Related to Integral Closure in $D[[x]]$* , Trans. Amer. Math. Soc., **122** Issue 2 (1966), 321–333.
- [21] B. Olberding, *Globalizing local properties of Prüfer domains*, J. Algebra **205** (1998), 480–504.
- [22] B. Olberding, *Stability, duality and 2-generated ideals, and a canonical decomposition of modules*, Rend. Semin. Mat. Univ. Padova **106** (2001), 261–290.
- [23] B. Olberding, *On the classification of stable domains*, J. Algebra **243** (2001), 177–197.
- [24] B. Olberding, *On the structure of stable domains*, Comm. Algebra **30** (2002), 877–895.
- [25] B. Olberding, *Noetherian rings without finite normalizations*, Progress in commutative algebra 2, 171–203, W. de Gruyter, Berlin, 2012.
- [26] B. Olberding, *One-dimensional bad Noetherian domains*, Trans. Amer. Math. Soc. **366** (2014), 4067–4095.
- [27] B. Olberding, *Finitely stable rings*, Commutative Algebra - Recent Advances in Commutative Rings, Integer-valued Polynomials, and Polynomial functions, 269–291, Springer, New York, 2014.
- [28] B. Olberding, *One-dimensional stable rings*, submitted.

- [29] R. Remmert, *Classical topics in complex function theory*, Springer-Verlag, New York, 1998.
- [30] D.E. Rush, *Rings with two-generated ideals* J. Pure Appl. Algebra **73** (1991), 257–275.
- [31] D.E. Rush, *Two-generated ideals and representations of abelian groups over valuation rings*, J. Algebra **177** (1995), 77–101.
- [32] J. D. Sally and W. V. Vasconcelos, *Stable rings and a problem of Bass*, Bull. Amer. Math. Soc. **79** (1973), 574–576.
- [33] J. D. Sally and W. V. Vasconcelos, *Stable rings*, J. Pure Appl. Algebra **4** (1974), 319–336.
- [34] M. Zafrullah, *Completely integrally closed Prüfer v -multiplication domains*, preprint.

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